"BAYESIAN ESTIMATION TO SOME RELIABILITY AND SURVIVAL MODELS"

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DECLARATION

The thesis entitled "Bayesian Estimation of Some Reliability and Survival Models". Submitted to Bundelkhand University by me for the award of degree of Doctor of Philosophy is based on my research work carried on under the supervision of Professor V. K. Sehgal, Department of Mathematical Sciences and Computer Applications, Bundelkhand University, Jhansi.

I declare that this research work either in part or in full has not been submitted to any University or Institute for award of any degree.

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CERTIFICATE

Certified that the Thesis titled **"BAYESIAN ESTIMATION TO SOME RELIABILITY AND SURVIVAL MODELS"** submitted by Mr. CHANDRAPAL SINGH YADAV for the award of the Degree of DOCTOR OF PHILOSOPHY in MATHEMATICS to the BUNDELKHAND UNIVERSITY, JHANSI is a reward of his original work done under my supervision and has not been submitted elsewhere for a degree or Diploma in any form.

This is further certified that he has put in the required attendance.

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CHAPTER - 1

INTRODUCTION AND BASIC CONCEPTS

1.1 Introduction:

In present study we consider Baysian analysis to software reliability models. The most important feature of using Baysian model is that prior information can be incorporated into the estimation procedure. Baysian methods are especially useful in reliability analysis because of the increase in reliability is usually achieved by improving a similar product developed previously. For software systems there is always some available – information about the software development and useful information can also be obtained from similar software products through for examples. Some measures of the software complexity. The fault removal history during the design phase etc. with an appropriate prior, Baysian inference is quite accurate and gives much better results than other methods such as method of maximum likelihood or least squares. Also Baysian method requires fewer test data to achieve a high accuracy of estimation. It shall be noted that care must be taken in choosing appropriate prior distributions

because an improper prior distribution may give very bad results which can even be worse than the maximum likelihood estimates. However, if no information seems to be available, non-informative priors can still be used. They usually give quite adequate results. However, existing models seem to be monotonic in a sense that simple and mathematically tractable posterior distributions are few in the existing literature. Lose of the existing Baysian models are also strangely related to the JM (Jelinski-Moranda) Model developed by Jelinski and Moranda (1972), which is the simplest pure software reliability model and it seems quite popular, statisticians see the models discussed in Abdel-Ggaly et. al. (1986) and Mazzuchi and Soyer (1988). It is argued that Baysian estimates of the parameters in the JM–Model usually have better behaviours than the maximum likelihood estimates and this is the reason why many such models are studied. Most existing Baysaan formulation of the JM–Model assume in a sense that the failure intensity per fault follows a Gamma distribution and the number of initial faults is Poison distributed. These assumptions are used mainly due to the analytical simplicity of the posterior distributions. There is a need to search for other suitable aprior distributions in order to describe other aprior information. A recent paper by Becker and Comarinopoulous (1990) provides interesting ideas. Given

the values of λ ., the time to the first failure is exponential with parameter λ . The prior distribution of his assumed to belong to the following class $f(\lambda) = e^{-b\lambda} \sum_{i=0}^{n} a_i \lambda_i$. It is a class of conjugate priors with some desirable properties. The original model is one which takes the possibility of a correct program into account and can be seem as a generalisation of the model by Thompson and Chelson (1980). Due to the computational complexity, it is important that posterior distributions are not too complicated. There are also some computational procedures developed see e.g. Mazzuchi and Soyer (1988). However as for the Langberg – Singhpurwallia's general Baysian model, the modelling procedures are essentially the same. The earliest paper presenting a Baysian Model is due to Littlewood and Verrall (1973). In the papers of Littlewood (1979a, 1980 a) and (1980 b) where some other modifications may be found. Significant steps towards pure Baysian analysis of software reliability models have been taken by Singpurwallia and his colleagues Interesting papers are Meinhold and Singpurwallia (1983), Craw and Singpurwallia (1984) Horigome et al (1984), Bahlow and Singpurwallia (1985) and Langberg and Singpurwallia (1985). There are many papers dealing with Baysian treatment of the Jelinski – Moranda model. Important papers are Jewell (1985), Littlewood and Sofer (1987), Raftery (1988) and Csenki

(1990). Other interesting papers Presenting or Discussing Baysian software reliability models are Thompson and Chelson (1980), Musa (1984), Kyparisis and Singpurwallia (1984), Ross (1985 a), Abdel – Ghaly et al (1986), Langberg and Singpurwallia (1986), Jewell (1986) Liu (1987), Mazzuchi and Soyer (1988), Catuneanu and Mihalache (1985), Horigome and Kaise (1990), Becker and Camarinopoulous (1990) and Bunday and Al-Ayobui (1990)

1.2 A Brief Sketch of Statistical Techniques Used

1.2.1 An Outline of Baysian Inference

The foundation stone of this technique is 'Bayes' theorem and conditional probability. This theorem was first presented by Reverend Thomas Bayes (1963) and English minister. Laplace modified 'Bayes' basic theorem and the modified version is known as Bayes' theorem. Some of the standard texts giving more detailed discussions are those of Lindley (1965) and Box and Tiao (1973). For the basic theory and foundations one can refer to Sagave (1962) and Jeffreys (1961). Bayes' analysis is an essentially self-contained paradigm for statistics. It is an excellent alternative to use of large-sample asymptotic statistical

procedures. Bayesian procedures are almost always equivalent to classical large sample procedure when the sample size is very large.

In the parametric inference, the form of the population $f(x,\lambda)$ is known while the parameter λ known as labelling parameter, real valued or vector valued as the case may be, is unknown. However, we agree upon the parameter space i.e., the set of all possible values of the parameter which we denote by Ω. More so, this parameter λ can also be represented by mean lifetime or failure rate or other such quantities associated with the underlying population from which the sample has been obtained.

For example – the lifetime of a system follow exponential distribution with p.d.f. $f(x,\lambda) = \frac{1}{\lambda} e^{-x/\lambda}; x > 0, \lambda > 0$ .... (1.1)

λ here stands for mean lifetime and the parameter space is $\Omega = \{\lambda \mid \lambda > 0\}$

In classical estimation theory the estimate of λ depends only on sample values which we draw from $f(x,\lambda)$ and as such the information about λ from other sources as well. If this additional information is in the from of a probability distribution of λ ., then it can be combined with data using Bayesian framework. There are cases in which λ can be

regarded as a r.v. with p.d.f. g(λ). For example in case of exponential model, the mean life λ may be regard as varying from batch to batch over time and this variation may be represented by a probability distribution over Ω. As another example, suppose that woman has n sons, of whom x are haemophiliac (n–x) are normal. The probability of this event is

$$p(x,\lambda) = \binom{n}{x}\lambda^x(1-\lambda)^{n-x} \qquad \text{......... (1.2)}$$

Where λ is the probability that a particular sons will be haemophiliac. The problem is to make inferences about X. Given no additional information about λ, inferences would be based on (1.2) However, it may be possible to extract some information about λ by examining the woman's family tree. For instance, suppose that the woman had normal parents but she had a brother who was hemopbilic, then her mother must have been a carrier of the disease and therefore, she had a 50% chance of inheriting the gene for hemophilic. If she did inherit the gene, then there is a 50% chance that a particular son inherit the disease (λ =½) and if she did not, all of her sons will be normal (λ =0). The prior probability distribution of λ thus, becomes

$$\left.\begin{aligned} g(0) &= p[\lambda = 0] = \frac{1}{2} \\ g(\tfrac{1}{2}) &= p[\lambda = \tfrac{1}{2}] = \frac{1}{2} \end{aligned}\right\} \qquad \text{.........(1.3)}$$

Thus the set up can be understood as follows:

Suppose that n items are placed on a test. It is that their recorded life times form a random sample say $x_1, x_2 \ldots\ldots x_n$, which follow a distribution with p.d.f. $f(x, \lambda)$. To be specific we will assume λ to be real valued. Consider λ itself as a r.v. with p.d.f. $g(\lambda)$. Thus, the failure time p.d.f. $f(x,\lambda)$ can be regarded as a conditional p.d.f. of x given λ.

i.e. $f(x|\lambda)$ where the marginal p.d.f. of λ is given by $g(\lambda)$. Therefore

the joint p.d.f. of $(x_1, x_2 \ldots\ldots x_n, \lambda)$ is expressed as $H(x_1 x_2 \ldots\ldots\ldots\ldots x_n|\lambda)$

$$= \prod^n f(x_i|\lambda)g(\lambda) = L(x_1 x_2 \ldots\ldots\ldots\ldots x_n|\lambda) \qquad \ldots\ldots\ldots (1.4)$$

The marginal p.d.f. of $(x_1, x_2 \ldots\ldots x_n)$. is given by

$$P(x_1, x_2 \ldots\ldots x_n) = \int_\Omega H(x_1, x_2 \ldots\ldots x_n|\lambda)d(\lambda) \qquad \ldots\ldots\ldots (1.5)$$

Thus the conditional p.d.f. of λ given data $(x_1, x_2 \ldots x_n)$ is given by

$$\prod(\lambda|x_1, x_2 \ldots\ldots x_n) = \frac{H(x_1, x_2 \ldots\ldots x_n|\lambda)}{P(x_1, x_2 \ldots\ldots x_n)} = \frac{L(x_1, x_2 \ldots\ldots x_n|\lambda)g(\lambda)}{\int L(x_1, x_2 \ldots\ldots x_n|\lambda)g(\lambda)d\lambda} \ldots\ldots (1.6)$$

Thus, prior to obtaining the data (x_1, x_2 .....x_n), the variations in λ were represented by $g(\lambda)$ known as the prior distribution on λ. However, after the data (x_1, x_2 .....x_n) has been observed, in the light of new information, the variations in λ are represented by $\Pi(\lambda|x_1, x_2.....x_n)$, the posterior distribution of λ the uncertainty about the parameter λ prior to the experiment is by the prior p.d.f. $g(\lambda)$ and the same after the experiment is represented by the posterior p.d.f. $\Pi(\lambda|x_1, x_2.....x_n)$ in ....(1.6)

This process is a straight forward of Bayes theorem. After obtaining the posterior distributions of the parameters involved in the parent population, any statistical inference like estimation, testing of hypothesis about these parameters may be drawn with the help of this distribution.

In Bayes approach, the estimator of λ i.e., λ^* is one which minimizes expected loss w.r.t. the posterior distribution i.e., it depends on the loss function chosen. If the loss function is taken as quadratic loss function defined as $L(\lambda^*, \lambda) = (\lambda^* - \lambda)^2$ then the Bayes estimator that accomplishes the task of estimating λ is the posterior mean, i.e.,

$$\lambda^* = E[\lambda|x_1, x_2.....x_n] = \int_\Omega \lambda \, \Pi(\lambda|x_1, x_2.....x_n) \; d\lambda \qquad .... (1.7)$$

Using the posterior distribution, a $(1-\alpha)$ 100% Bayes' confidence interval p_1, p_2 for λ may be obtained from =

$$\int_{p_1}^{p^2/} \Pi(\lambda|x_1, x_2 \ldots x_n) \; d\lambda = 1-\alpha \qquad \ldots\ldots\ldots (1.8)$$

1.2.2 Concept of Prior and Its Definition

A detailed discussion to obtain the solution to the problem concerning the choice of a prior distribution of λ is given in Raffia and Schlaifer (1961) but here we shall confine overselves just in defining them priors for the parameters in distributions differ in respect of their hidden properties.

A family of priors is said to be **conjugate**, if it is 'closed under sampling [Lindly, 1972]. If the same family, then this family is said to have 'closure under sampling' property [Weitherill, 1961]. Faiffa and Schlaifer, 1961] have considered a method of gathering prior densities on the parameter space. A family of such densities has been called by them a 'natural conjugate family'. For example in case of an exponential density, the inverted gamma priors from such a family.

Jeffreys (1961) proposed a general rule for obtaining the prior distribution on the unknown parameter λ. According to this rule, λ which is assumed to be a r.v. follows such a distribution which is proportional to the square root of the Fisher information on λ.

Mathematically, $g(\lambda)\alpha\frac{1}{\sqrt{I(\lambda)}}$ or $g(\lambda)$ = constant $\sqrt{I(\lambda)}$ where,

$$I(\lambda) = E\left[\left\{\frac{\partial \log L(x|\lambda)}{\partial\lambda}\right\}^2\right] \underline{\underline{E}}\left[\left\{\frac{\partial^2 \log L(x|\lambda)}{\partial\lambda^2}\right\}\right]$$

A difficulty arises when the prior information about the parameter is rather 'vague' or worse still there is no prior information whatever. This leads to the consideration of what are known as improper or quasi prior distribution. For a proper prior we have $g(\lambda)\geq 0$ and $\int_\Omega g(\lambda)d\lambda = 1$ while for an improper prior, $g(\lambda)\geq 0$ but $\int_\Omega g(\lambda)d\lambda \neq 1$

Jeffreys prior may or may not be proper. Various rules have also been suggested for the selection of a prior but no neat solution appears to the problem till now.

1.2.3 Non-Parametric or Distribution–Free Methods

In life testing experiment it is not always practically possible to formulate a statistical model for failure time. Specially, at the early stage of designing a system, a designer knows little about the functional performance of the system. In such situations it becomes, important to develop distribution free techniques for the assessment of the reliability characteristics of a system. Kaplan and Meier (1958) suggested estimate for survival function based on an empirical study using censored data. Some other non-parametric methods have been discussed by Breslow and Crowley (1974), Peterson (1977), Johansen (1978) and Sharma and Krishna (1993).

The most rewarding feature of non-parametric models is that we do not lose much in terms of answering the typical questions of concern in reliability studies, as compared to the more restrictive parametric models. Besides the obvious inherent advantage of the non-parametric methods, we also have the nice property of relative insensitivity of outliers in the data, superior power properties for a wide class of alternative distributions, and using the test statistics we can obtain estimators and distribution free confidence intervals for the population parameters of interest.

In fact, since F_n, (x) is the sample mean of r.v's $I(-\infty, x)$ (xi), ...I $(-\infty, x)$ (x_n), we know by the central-limit theorem that $F_n(x)$ is asymptotically normally distributed with mean F(x) and variance $\frac{1}{n}F(x)[1-F(x)]$. Equation (1.10) and (1.11) show that for fixed x, $F_n(x)$ is an unbiased and mean-squared error consistent estimator of F(x) regardless of the form of F(.).

1.3 Sequential Procedures

The first formalised sequential procedure is the 'double sample' inspection plan by Dodge and Roming (1929) which was subsequently generalised to a multistage procedure by Bartky (1943). Although there were other suggestion of multi-stage procedures (Hotelling, 1943), a systematic development of sequential methods started only in the last two years of World War II independently in USA and Great Britain. Abraham Wald developed a general theory of sequential analysis and in particular proposed the 'Sequential Probability Ratio Test' (SPRT) taking cue from a problem posed by the Statistical Research Group at Columbia University in 1943, Works of Wald (1947), Wald and Wolfwitz (1948), Arrow, Blackwell and Girshick (1949) and their co-workers laid the foundation of sequential decision procedures.

A procedure of making inferences about the distribution of one or more variable (r.v) is called a sequential procedure. The principal feature of such a procedure is a sampling scheme which lays down a rule under which one decides at each stage of the sampling whether to stop or to continue sampling, this decision being taken in the light of the observations already obtained. A general sequential procedure has two aspects first as a stopping rule, which tells us when to stop sampling and, second as an action rule which tells us what type of inference to make after sampling has been stopped.

In order to judge the relative merits of two or more rival sequential tests, we make use of two criteria, one being the operating characteristic (OC) function and other the average sample number (ASN) function. The OC function often called power of sequential test, denote by $L(\theta)$ is defined as the probability of accepting the null hypothesis when θ is the true value of the parameter. It is closely related to the notion of the power function in the fixed sample size testing procedures. Now, since the number of observations say N, required by a sequential procedure to reach a decision is not predetermined but is a r.v. If we carry out the same sequential procedure repeatedly, we shall get different values of N.

same sequential procedure repeatedly, we shall get different values of N. We may call a procedure preferable if it requires a small value of N on the average. This average value of N is called the average sample number of the sequential procedure. Thus, the OC function describes how well the procedure achieves objective of making correct decision, while ASN function represents the price one has to pay to reach a decision, in terms of the number of observations required by the test.

1.3.1 The Sequential Probability Ratio Test (SPRT)

Let x_1, x_2 .... be a sequence of identically and independently distributed (iid) random variables with common density (mass) function $f(x,\theta)$. Let us consider the testing of a simple hypothesis $H_0 : \theta = \theta_0$ against a simple alternative, $H_1 : \theta = \theta_1$ when the observations are taken sequentially.

Let f_{in} denote the joint p.d.f's (p.m.f.s.) of x_1, x_2 ...., x_n under $H_i (i = 0,1)$

Define, $\lambda_n(\underline{x}) = \dfrac{f_{in}(x)}{f_{on}(x)}$; $\underline{x} = (x_1, x_2 ......x_n)$ .... (1.12)

Then, the SPRT for testing H_0 vs H_1 is a rule which states :

(i) Accept H_0 and terminate the experiment, if $\lambda_n(\underline{x}) \leq B$

(ii) Reject H_0 and terminate the experiment, if $\lambda_n(\underline{x}) \leq B$

(iii) The experiment is continued by taking an additional observation if $B < \lambda_n(\underline{x}) < A$

Here A and B (A > B) are the two positive constant which are determined so that test will have strength (α,β) where α,β are the probability of type I error and type II error respectively. If N is the stopping r.v., then

$$\alpha = P_{\theta_0}(\lambda_n(\underline{x}) \geq A) \qquad \text{........ (1.12)}$$

and $\beta = P_{\theta_0}(\lambda_n(\underline{x}) \leq A)$ .... (1.13)

Here, $A = \dfrac{1-\beta}{\alpha}$ and $B = \dfrac{\beta}{1-\alpha}$

Efficiency of SPRT

By the efficiency of the SPRT, say e_i we mean the ratio

$$e_i = \frac{N_i(\alpha,\beta)}{E\theta_i(N)}; i = 0,1 \qquad \text{........ (1.14)}$$

Here, $N_i(\alpha,\beta)$ is the minimum value of $E_{\theta i}(N)$ in the class (i=0,1) clearly, $0 < e_i < 1$. An optimum test has the efficiency one under H_0 as well as under H_1. Wald (1947) has shown that the efficiency of the SPRT is very near to one under H_0 as well as under H_1.

1.3.2 Robustness

Most of the statistical Inference problems are parametric in nature. We have assumed that the functional form of the distribution being sampled is known except for a finite number of parameters. It is be expected that any estimate or test of hypothesis concerning the unknown parameters constructed on this assumption will perform better than the corresponding non-parametric procedure, provided that the underlying assumptions are satisfied. It is therefore of interest to know well the parametric optimal tests or estimates constructed for one population perform when the basic assumptions are modified. If we can construct tests or estimates that perform well for a variety of distributions, for example, there would be little point in using the corresponding nonparametric method unless the assumptions are seriously violated.

In practice, one makes many assumptions in parametric inference, and any one or all of these may be violated. Thus one seldom has accurate knowledge about the true underlying distribution. Similarly, the assumption of mutual independence or even indentical distribution may not hold. Any test or estimate that performs well under modifications of underlying assumptions is usually referred to as robust.

The subject of robustness is receiving considerable attention of late. The first theoretic approach to the robust statistics was introduced by Huber (1964). His book on this topic Huber (1981) made this fundamental work to a wider audience. A different theoretic approach to robust statistics was originated by Hampel (1968, 1971, 1974). Robust statistics, in a loose, non-technical sense, is concerns with the fact that assumptions commonly made in statistics are at most approximations to reality. As a collection of related theories, robust statistics is the statistics of the approximate parametric methods. The moral is clear; One should check carefully to see that the underlying assumptions are satisfied before using parametric methods.

1.3.3 Statistical Quality Control (Or Inspection) : A Way to Increase Reliability

How can it happen that poor quality, unreliable products can be put out by a plant and then used by the consumer? In the first place, many of the properties of the unit that constitute its quality are either very difficult to check under factory conditions or are simply impossible to be checked because of the destructive nature of tests. In the second place the poor quality may be the result of the fact that the parameters

characterizing the operation of the units are unstable, During the testing period, their values are completely acceptable to the user but then, after they stored or used for a while, they change in an erratic manner.

Tests on durability and reliability are usually of destructive nature. However, inspection may be either destructive or constructive. Since, testing the stability of parameters under different environmental conditions is of great significance in reliability theory, inspection is usually destructive or, in any case, it changes the quality of the units. These environmental conditions greatly affect the working of the engineering systems e.g., the debugging or bum-in period generally results in decreasing failure-rate and the wear-out period resulting due to aging-phenomena causes irreversible changes in the parameters or the unit leading to increasing failure rate. The failure – phenomena of an individual or a system is treated to be under normal operation when a constant or near constant failure–rate pattern is observed. After having an eye view of different failure-time models discussed earlier and in view of above, we can say that parameters in failure time distributions represent these environmental conditions.

Statistical Inspection (or S.Q.C.) problems are most closely related to the problem of ensuring reliability. Everything depends on the specific forms of the inspection operation. If a failure can occur during the inspection operation, S.Q.C. at the same time becomes a control over reliability. However, statistical inspection problems are more general. But properly restricted statistical inspection is a means of increasing the reliability of units, Obviously, during the course of inspections, testing techniques in statistical inference become the basis of decision making procedures on the disposal of allot of devices subject to constraints on risks (producer's and consumer's) and cost.

1.4 Posterior Analysis of the Mixture of Hazard-Rate Model

Engineering Systems, while in operation, are always subject to environmental stresses and shocks which may or may not alter the failure pattern of the system. Suppose, $p(0<p<1)$ is the unknown probability that the system usable to bear these stresses and its failure pattern remains unaffected, and also $q(=1-p)$ is the probability that the system does not bear these stresses and the failure pattern of the system undergoes a change. In such situations, a failure distribution is generally used to describe mathematically the failure pattern of the system. To some extent,

the solution to the proposed problem is attempted through the mixture of distributions. To deal with such type of problem, Mendenhall and Hader (1958) have considered a failure population which may be divided into subpopulations, each subpopulation being exponentially distributed. The m.l.e's for the population parameter were obtained from samples censored at a predetermined test termination time. Kao (1959) suggested the use of a mixture of two Weibull distribution as a lifetime model in which one of the components (Weibull) represents sudden (or catastrophic) failure and the other component (Weibull) represents wearout (or delayed) failure in electronic tubes. Proschan (1963) explored the theoretical basis for a non-increasing failure rate and applied the results to real data from a mixture of exponentially distributed sub-populations. Kleyele and Dahiya (1975) studied a mixture of binomial and exponential population from censored data. The estimation of the parameters of a mixture of two distributions from a general parametric family of distributions assuming an ordering of the parameters was the subject matter of Shaked and Tran (1982). Papadopoulos and Padgett (1985) studied mixed failure population in which each subpopulation was exponentially distributed MLE's as well as Bayes estimators for the population parameters and the reliability function were obtained from

right censored samples. The Inverse Gaussina-Weibull mixture model was proposed as a life-time model by AL-Hussaini and Abd-El-Hakim (1989), in which each component represents a different type of failure such as the sudden or the wearout failure. It has been found that the use of the mixture model is essential in many situations to meet the relevant conclusions. It is mainly used to carry out analysis of weak models. The fertility analysis in survival studies is more adequate with the mixture model rather than the simple model. An interesting situation for using the mixture model is the analysis of there Reliability of a bulb knowing that the bulbs are being supplied by a fake company also with the original brand in the market. In this particular situation simple reliability can not help the analyst to reach a reliable conclusion. The additional advantage of the mixture model is that it helps in dealing with the more practical situations in the survival analysis. In most of the situations, while studying through a failure model we assume that the output is equally reliable for whole articles produced, while the guiding factor, i.e. machine or system may differ in context to produce the assumed articles. It is natural to say that all machines or systems in use may not produce the articles with equal reliableness and quality. Obviously, a new machine will produce a more reliable item than an older one.

A lot of above discussion about the mixture model reveal that generally a failure distribution represents an attempt to describe mathematically, the life-time r.v, and failure data may reveal a number of physical causes responsible for the system failure. However, in this regard we face with two types of problems. Firstly, there are many physical causes that individually or collectively cause the failure of the system or device. Presently, it is not possible to differentiate between these physical causes and mathematically account for all of them, and therefore, this in-isolation of the causes creates problem regarding the choice of a failure distribution. Secondly, even if a goodness of fit technique is applied to actual observations of time to failure, we face a problem a risint due to the non-symmetric nature of the life-time distributions whose behaviour is quite different at the tails where actual observations are appears inview of the limited sample size (Mann, Schaffer and Singpurwalla 1974). Obviously, the best one can do is to lookout for a concept which is useful for differentiating between different lifetime distributions. "Hazard-rate" which is the measure of instantaneous speed of failure, is one such concept in the literature on reliability. After analysing such physical considerations of the system, we can formulate a mixture of hazard–rate function which, in turn, provide the failure time distribution.

In view of the above, and due to continuous stresses and shocks on the system, let us suppose that the hazard-rate pattern of a with probability q. let the hazard-rate pattern of the system in these two situations be in either of the following two states.

State 1 : Initially it experiences a constant hazard-rate (or chance failure) pattern and this pattern may (or may not) change with probability $q(p=1-q)$.

State 2 : If the stresses and shocks alter the hazard-rate pattern of the system with probability q. then it experiences a wear-out failure pattern.

Using such a hazard-rate pattern, the characterisation of life-time distribution in the corresponding situations given. Various inferential properties of this life-time distribution along with the estimation of parameters and reliability characteristics is the subject matter of the study. Since, the estimates based on the operational data can be updated by incorporating past environmental experiences on the random variations in the life-time parameters [Martz and Waller 1982]. Therefore, the Bayesian

analysis of the parameters and other reliability characteristics is also given. Results have been highlighted with suitable example to show the application aspect of the developed procedures.

1.5 Statistical Background

Let, p: the probability of the event A, that the system is able to bear the stresses and shocks and its failure pattern remains unaltered.

q = 1–p: the probability of the event complementary to A, i.e. $\overline{A}$ further, let, the mixture of the hazard-rate function be

$$h(t) = p\lambda + (1-p)\lambda t; \lambda, t > 0, 0 \le p \le 1 \qquad \text{........ (1.15)}$$

In particularly, if,

(i) p = 1, then the hazard-rate function h(t) = λ and it represents the hazard-rate of an exponential distribution.

(ii) p=0, then h(t) = λ t and it represents the hazard-rate of the Rayleight distributioner Weibull distribution with its shape parameter as 2.

1.5.1 Characterisation of the Life-Time Distribution

Using, the well-known relationship $f(t) = h\exp\left[\int_0^t (x)dx\right]$ and inview of (1.15), the p.d.f. of the life-time T can be obtained as

$$f(t) = (p\lambda + (1-p)\lambda t) \;\exp\left[-\int_0^t \{p\lambda + (1-\pi)\lambda x\} dx\right]$$

$$= [p\lambda + (1-p)\lambda t] \ \exp[-\{p\lambda t + \tfrac{1}{2}(1-p)\lambda t^2\}; \lambda, t > 0 \qquad \ldots\ldots\ldots (1.16)$$

= 0, otherwise,

the reliability function of the system is given by

$$R(t) = \frac{f(t)}{h(t)} = \exp[-\{p\lambda t + \frac{1}{2}(1-p)\lambda l^2\}]; t > 0 \qquad \ldots\ldots\ldots (1.17)$$

and the mean time to system failure (MTSF) can be evaluated as

$$\text{MTSF} = E(T) = \int_0^\infty R(t)dt = \int_0^\infty \exp[-\{p\lambda t + \frac{1}{2}(1-p)\lambda t^2\}]dt$$

$$= \exp\left[\frac{p^2\lambda}{2(1-p)}\right]\int_0^\infty \exp\left[-\frac{\lambda(1-p)}{2}\left\{t + \frac{p}{1-p}\right\}^2\right] dt \qquad \ldots\ldots\ldots (1.18)$$

on making the transformation $z = \frac{\lambda(1-p)}{2}\left[t + \frac{p}{1-p)}\right]^2$ in (1.18) and

$dz = \lambda(1-p)\left[t + \frac{p}{(1-p)}\right]dt$, one gets

$$MTSF = \frac{1}{\sqrt{2\lambda(1-p)}} \exp\left[\frac{p^2\lambda}{2(1-p)}\right]\int_{\frac{p^2\lambda}{2(1-p)}}^\infty z^{1/2-1}e^{-z} \ dz$$

$$= \sqrt{\left(\frac{\pi}{2\lambda(1-p)}\right)} \exp\left[\frac{p^2\lambda}{2(1-p)}\right]\left[1 - \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^{\frac{p^2\lambda}{2(1-p)}} z^{\frac{1}{2}-1} e^{-z}dz\right]$$

$$= \sqrt{\left(\frac{\pi}{2\lambda(1-p)}\right)} \exp\left[\frac{p^2\lambda}{2(1-p)}\right] \left[1 - I_{1/2}\left\{\frac{p^2\lambda}{2(1-p)}\right\}\right] \qquad \ldots\ldots\ldots (1.19)$$

where $I_m(x)\frac{1}{m}\int_0^x z^{m-1}e^{-z}dz$ is the well known incomplete gamma function and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

1.5.2 Classical Estimation of Parameters and Reliability Characteristics

Let, $t_1, t_2......, t_n$ be the random failure times of n items under test whose failure time distribution is as given in (1.16). Then, the likelihood function of the sample is given by

$$L(t_1,t_2......,t_n|\lambda,p)=\lambda^2\left[\prod_{i=1}^{n}\{p+(1-p)t_i\}\right]\ \exp.\left[-\lambda\sum_{i=1}^{n}\left\{pt_i+\frac{1}{2}(1-p)t_i^2\right\}\right] \text{....(1.21)}$$

1.5.3.1 (MLE's) Maximum Likelihood Estimates

Using the likelihood function is (1.21) the estimates of various parameters involved in the p.d.f. given in (1.21) can be obtained as follows:-

Case I : When p is known

Taking logarithm on both sides of (1.21), we have

$$\log L(t_1,t_2......,t_n|\lambda,p)=n\log\lambda+\sum_{i=1}^{n}[p+(1-p)t_i]-\lambda\sum_{i=1}^{n}\left[pt_i^2+\frac{1}{2}(1-p)t_i^2\right] \text{....(1.22)}$$

Now, to obtain the m.l.e. of λ, say $\hat{\lambda}$, we consider

$$\frac{\partial\log L(t_1,t_2.....,t_n|\lambda,p)}{\partial\lambda}=0 \text{ which gives, } \hat{\lambda}=\frac{n}{\sum_{i=1}^{n}\left[pt_i+\frac{1}{2}(1-p)t_i^2\right]} \text{ ..... (1.23)}$$

On using the invariance property of m.l.e's, the m.l.e. of R(t), h(t) and MTSF respectively given in (1.22), (1.26) and (1.23) may be written as:-

(i) The m.l.e. for R(t), say $\hat{R}_1(t)$, will be

$$\hat{R}_1(t) = \exp\left[-\hat{\lambda}\left\{pt + \frac{1}{2}(1-p)t^2\right\}\right] \quad \text{..... (1.24)}$$

(ii) The m.l.e. h(t), say $\hat{h}_1(t)$, will be $\hat{h}_1(t) = \hat{\lambda}[p + (1-p)t]$ ..... (1.25)

(iii) The m.l.e. for MTSF will be

$$M\hat{T}SF = \sqrt{\left(\frac{\pi}{2\hat{\lambda}(1-p)}\right)}\exp\left[\frac{p^2\hat{\lambda}^2}{2(1-p)}1 - I_{1/2}\left\{\frac{p^2\hat{\lambda}^2}{2(1-p)}\right\}\right] \quad \text{..... (1.26)}$$

Case II : When λ is known

To find the m.l.e. of p, say $\hat{p}$, we consider

$$\frac{\partial \log L(t_1, t_2, \ldots, t_n | \lambda, p)}{\partial p} = 0$$

$$\sum_{i=}^{n}\left[\frac{(1-t_i)}{p+(1-p)t_i}\right] - \lambda\sum_{i=}^{n}\left[t_i - \frac{1}{2}t_i^2\right] = 0$$

or $$\sum_{i=1}^{n}\left[\frac{1}{p+\frac{t_i}{(1-t_i)}}\right] - \frac{\lambda}{2}\sum_{i=1}^{n}t_i(2-t_i) = 0 \quad \text{..... (1.27)}$$

Estimate of p, i.e., $\hat{p}$, can be obtained from (1.27) by using Newton-Raphson or other suitable iterative or inverse interpolation method.

Again on using the invariance property of m.l.e's, one gets

(i) The m.l.e. for R(t), say $\hat{R}_2(t)$, as $\hat{R}_2(t) = \exp\left[-\lambda\left\{\hat{p}t + \frac{1}{2}(1-\hat{p})t^2\right\}\right]$ .. (1.28)

(ii) The m.l.e. for h(t), say $\hat{h}_2(t)$, as $\hat{h}_2(t) = \lambda[\hat{p} + (1-\hat{p})t]$ ..... (1.29)

Case III : When λ and p both are unknown

Similarly, the simultaneous estimate of λ and p are solutions of the equations

or
$$\hat{\lambda} = \frac{n}{\sum_{i=}^{n}\left[\hat{p}t_i + \frac{1}{2}(1-\hat{p})t_i^2\right]} \qquad \text{..... (1.30)}$$

and
$$\sum_{i=1}^{n}\left[\frac{1}{\hat{p} + \frac{t_i}{(1-t_i)}}\right] = -\frac{\hat{\lambda}}{2}\sum_{i=1}^{n} t_i(2-t_i) = 0 \qquad \text{..... (1.31)}$$

(1.31) may be solved for $\hat{p}$ by Newton-Raphon or other suitable iterative method and this value is substituted in (1.29) to obtain $\hat{\lambda}$.

On using the invariance property of m.l.e.'s,

(i) The m.l.e. for R(t), say $\hat{R}_3(t)$ becomes

$$\hat{R}_3(t) = \exp\left[-\hat{\lambda}\left\{\hat{p}t + \frac{1}{2}(1-\hat{p})t^2\right\}\right] \qquad \text{..... (1.32)}$$

(ii) The m.l.e. for h(t), say $\hat{h}_3(t)$, becomes

$$\hat{h}_3(t) = \hat{\lambda}[\hat{p} + (1-\hat{p})t] \qquad \text{..... (1.33)}$$

1.5.4. Posterior Analysis

Case I : When p is known

Considering the conjugate prior distribution of λ to be gamma with p.d.f. $g_2(\lambda) = \frac{\lambda^{\beta}}{\Gamma(\beta)} \lambda^{\beta-1} e^{-\alpha\lambda}; \alpha, \beta, \lambda > 0$ ..... (1.34)

The likelihood function in (1.16) may be rewritten as

$$L t_1, t_2 \ldots\ldots, t_n | \lambda, p) = \lambda^n T_1 e^{-\lambda T_2} \qquad \text{..... (1.35)}$$

Where, $T_1 = \prod_{i=}^{v} [p + (1-p)t_i]$ ..... (1.36)

and $T_2 = \sum_{i=}^{n} \left[pt_i + \frac{1}{2}(1-p)t^2{}_i \right]$ ..... (1.37)

Inview of the prior distribution of λ in (1.34) and the likelihood function in (1.35), the posterior distribution of λ given t_1, t_2......, t_n may be derived as

$$\Pi_1(\lambda | t_1, t_2 \ldots\ldots, t_n p) = \frac{L(t_1, t_2 \ldots\ldots, t_n | \lambda, p) g_1(\lambda)}{\int_0^{\infty} L(t_1, t_2 \ldots\ldots, t_n | \lambda, p) g_1(\lambda) d\lambda}$$

$$= \lambda^{n+\beta-1} \frac{e^{-\lambda(\alpha+T_2)}}{\int_0^{\infty} \lambda^{n+\beta-1} e^{-\lambda(\alpha+T_2)} d\lambda}$$

$$= \frac{(\alpha + T_2)^{n+\beta}}{\Gamma(n+\beta)} \lambda^{n+\beta-1} e^{-\lambda(\alpha+T_2)}; (\alpha + T_2), (n+\beta), \lambda > 0 \qquad \text{..... (1.38)}$$

Which is a gamma distribution with parameters $(\lambda + T_2)$ and $(n+\beta)$.

Therefore, the Bayes estimate of λ, say λ^*, under the square-error loss function becomes,

$$\lambda^* = E[\lambda|t_1, t_2......, t_n p] = \int_0^\infty \lambda \Pi_1(\lambda|t_1, t_2....., t_n, p)d\lambda$$

$$= = \frac{(\alpha + T_2)^{n+\beta}}{\Gamma(n+\beta)} \int_0^\infty \lambda^{n+\beta} e^{-\lambda}(\alpha + T_2)d\lambda = \frac{n+\beta}{\alpha + T_2} \qquad ..... (1.39)$$

The corresponding posterior variance of λ is given by

$$V(\lambda|t_1, t_2....., t_n, p) = E(\lambda^2|t_1, t_2....., t_n, p) - [E(\lambda|t_1, t_2....., t_n, p)]^2 \qquad ..... (1.40)$$

Now, $E(\lambda^2|(t_1, t_2.....t_n, p) = \int_0^\infty \lambda^2 \Pi_1(\lambda|t_1, t_2...., t_3, p)d\lambda$

$$= \frac{(\alpha + T_2)^{n+\beta}}{\Gamma(n+\beta)} \int_0^\infty \lambda^{n+\beta+1} e^{-\lambda(\alpha+T_2)} d\lambda = = \frac{(n+\beta)(n+\beta+1)}{(\alpha + T_2)^2} \qquad ..... (1.41)$$

Using (1.39) and (1.41) in (1.40), the posterior variance of λ is

$$V(\lambda|t_1, t_2......, t_n, p) = \frac{(n+\beta)}{(\alpha + T_2)^2} \qquad ..... (1.42)$$

The Bayes estimate of R(t) is $R^*(t) = E[R(t)|t_1, t_2......., t_n, p]$

$$= \int_0^\infty e^{-\lambda T_3} \Pi_1(\lambda|t_1, t_2......, t_n, p)d\lambda$$

$$= \frac{(\alpha + T_2)^{n+\beta}}{\Gamma(n+\beta)} \int_0^\infty \lambda^{n+\beta-1} e^{-\lambda(T_2+T_3+\alpha)} d\lambda$$

$$= \frac{1}{\left(1 + \dfrac{T_3}{\alpha + T_2}\right)^{n+\beta}} \qquad ..... (1.43)$$

Where, $T_3 = \left[pt + (1-p)\frac{t^2}{2}\right]$ ..... (1.44)

Also, $E[R^2(t)|t_1, t_2....,t_n, p] = \int_0^\infty e^{-2\lambda T_3} \prod_1(\lambda|t_1, t_2.....,t_n, p)d\lambda$

$$= \frac{1}{\left(1 + \frac{2T_3}{\alpha + T_2}\right)^{n+\beta}} \quad ..... (1.45)$$

Therefore, the posterior variance of reliability function R(t) is given by $V[R(t)|t_1, t_2....., t_n, p] = E[R^2(t)|t_1, t_2....., t_n, p] - [E(R(t)|t_1, t_2....., t_n, p)]^2$

$$= \frac{1}{\left(1 + \frac{2T_3}{\alpha + T_2}\right)^{n+\beta}} - \frac{1}{\left(1 + \frac{T_3}{\alpha + T_2}\right)^{2(n+\beta)}} \quad ..... (1.46)$$

Similarly, the Bayes estimate of h(t), say, $h_1^*(t)$, is

$h_1^*(t) = E[t_1, t_2....., t_n, p]$

$$= [p + (1-p)t]\int_0^\infty \lambda \prod_1(\lambda|t_1, t_2...., t_n, p)d\lambda$$

$$= [p + (1-p)t].\lambda^*$$

$$= \frac{(n+\beta)[p + (1-p)t}{(\alpha + T_2)} \quad ..... (1.47)$$

and the posterior variance of h(t) is given by

$V[h(t)|t_1, t_2...., t_n, p] = V[\{p + (1-p)t\}\lambda|t_1, t_2...., t_n, p]$

$$= [p + (1-p)t]^2 V(\lambda|t_1, t_2....., t_n, p)$$

$$= \frac{[p + (1-p)t]^2(n+\beta)}{(\alpha + T_2)^2} \quad ..... (1.48)$$

Further once the posterior distribution of λ has been obtained, one we get the Bayesian confidence limits for λ and other reliability characteristics by making some suitable transformations. The interval (ρ_1, ρ_2) is said to be a $(1-\alpha)$ 100% Bayes' confidence interval of λ if

$$\int_{\rho_1}^{\infty} \Pi_1(\lambda | t_1, t_2 ....., t_n, p) d\lambda = 1-\alpha \qquad ..... (1.49)$$

An equal tail (central) $(1-\alpha)$ 100% Bayes' confidence interval (ρ_1, ρ_2), for λ is given by

$$\int_{0}^{\rho_1} \Pi_1(\lambda | t_1, t_2 ....., t_n, p) d\lambda = \frac{\alpha}{2} = \int_{\rho_2}^{\infty} \Pi_1(\lambda | t_1, t_2 ....., t_n, p) d\lambda \qquad ..... (1.50)$$

Class II : When λ is Known

Let, the prior distribution of p be Beta distribution with p.d.f.

$$g_2(p) = \frac{1}{B(a,b)} p^{a-1}(1-p)^{b-1}; a,b > 0, 0 < p < 1 \qquad ..... (1.51)$$

1.6 Distribution-Free Inferences on System Reliability

System reliability characteristics are analysed to achieve the preassigned reliability standards. As such, experiments are conducted to record failure time date which are used for analysing the systems in

terms of reliability function, increasing or decreasing hazard rate and mean time to system failure etc. Inferences on the reliability function are drawn in determining the worth of a system to accomplish and intended task. A vast literature dealing with this aspect is given in references like Kapur and Lamberson (1977), Lawless (1982), Martz and Wailer (1982), and Sinha (1986). In all such studies, an assumption about the form of the failure time distribution or the system or sub-system is made and it is observed that the reliability functions are usually complicated functions of the estimate of the failure time parameters. Also, at its early designing phase, it is not practically feasible to make assumptions about the failure time distribution of the system. More so, sometimes, it becomes necessary to form a system configuration in which the component may or may not be identical. In all such cases also it is difficult to state the lifetime distribution of the system. Thus, in many practical situations, it is increasingly difficult to obtain failure time distribution of the system even if some admissible assumptions about the failure time distribution of the components are available. To overcome this difficulty, Sharma and Krishna (1995) obtained non-parametric distribution of the reliability function for some basic system configuration. The distribution so obtained may be used to get expected system reliability along with variance

and higher moments. Further, since the life-testing experiments are usually costly and time consuming. Therefore, censored samples are used for the statistical treatment of data even at the cost of a loss in efficiency. In experiments using ccnsored failure information, one would like to get sufficient information regarding the number of failures with a reasonable sample size or time limit for drawing valid inferences with a desirable accuracy. Here, one would like to restrict the sample size, while experimenting with high cost items, and limit the period of experimentation when the cost increases with time. The situation become more complex to draw parametric inferences about the reliability characteristics of the system when this time truncated failure information is used. It is therefore desirable to devise some procedure that are free of assumptions of the concerning failure time distribution. Some of the recent studies like Sharma and Bhutani (1992 b) and Sharma and Krishna (1994) have touched upon this aspect to some extent. Study in Sharma and Bhutani (1992 b) deals with the analysis of bounds for system reliability to perfomr an optimization between the sample size and censoring time. The developments in this study are quite useful for making an economic trade-off between sample size and censoring time. It is observed that estimates for the mean and variance of lifetimes distribution of the system

are enough to define such bounds. Still, the study has certain limitations as the bounds used for making such estimates of sample size and censoring time are restrictive to a situation when the mission time is greater than the mean lifetime. Sharma and Krishna (1994) have used Kolmogrov-Smirnov statistic for defining confidence limits for unreliability function. These confidence limits are found useful to define non-parametric estimates for the sample size and censoring time needed for the realisation of a preassigned failure information.

Further, an engineering system may be better designed to meet the reliability objectives if the past environmental variations in the parameter involved in the distribution of a concerned variable are also incorporated with experimental data. In such a case, the non-parametric Bayesian analysis of the system reliability is of obvious importance.

The development of some simple distribution-free techniques useful for testing the significance of the system reliability for a given mission time or when a time truncated failure information is to be used. The test for testing the significance difference between the reliability functions of the two systems with common mission time and for testing the significance

of a specified difference between two reliability functions are also given. The Bayesian estimation of the system reliability with a suitable prior distribution has been discussed here.

1.7 Bayes Test Plans for Testing A Series Configuration

A good manufacturer for the acceptance purposes always keeps a constant watch on the quality of products he manufactures. Such alertness enables to specification of the purchaser. On the other hand a purchaser is also anxious to satisfy himself about the quality of products he accepts. In quality control texts the manufactures and the purchaser are referred to as producer and consumer respectively. In order to assess the quality of products, the acceptance inspection is usually based on sampling. All acceptance plans for the items which are destructive in nature must inevitably be done by sampling. In many other instances sampling inspection is used because the cost of 100% inspection is prohibitive. Moreover, when there are a great many similar items of product to be inspected. It has been found that if a scientifically designed sampling inspection plan provides protection to consumer against the acceptance of too much defective product and, on the other hand,

producer is protected against the rejection of too much good product. It is obvious that consumer and producer have completely opposite view points towards the selection of sampling plans, i.e., they have conflicing interests. Practically it has been observed that no sampling plan can give complete protection to producer and as well as consumer, i.e, a sampling inspection plan is likely to involve some amount of sampling error which with result in producing certain risks on the part of producer and consumer both. These risks will be needful in construction of an acceptance sampling plan. Further, variation in the quality of manufactured product in the repetitive process in industry is inevitable. This variation will intern affect the producer's and consumer's risks. Considering this fact, Brush (1986) presented the process of comparing Bayes acceptance sampling plans with classical sampling plans. The study highlighted the importance of using Bayes producer's risk (BPR) as a supplement to the analysis of the modified classical measure of producer's risk (CPR*). After normalizing the quality on the index scale as in Brush et. al (1981) and Hoadly (1981), Brush (1986) defined BPR and compared it with the CPR*. The study in Brush (1986), however, does not define Bayes consumer's risk (BCR) and the modified classical measure of consumer's risk (CCR*). The difficulty in defining them concerns with the appropriate

definition of bad quality needed for defining BCR and CCR*. Later, Sharma and Bhutani (1992 c) investigated an acceptance sampling plan with given producer's and consumer's quality specifications. In addition, the concept of BCR is analysed as a supplement to CCR*.

Further, various engineering systems are analysed in respect of their reliability characteristics to meet the established reliability standards. It has been felt that the reliability characteristics of the system can be better analysed if the past environmental variations in the lifetime distribution of the system are also in the analysis. Similar to product control problems in the statistical quality control (SOC), systems are tested for their reliability characteristics like. MTSF, reliability, availability and hazard rate etc. Likewise, acceptance sampling plans need to be developed for accepting or rejecting a lot of system on the basis of certain reliability specifications when the parameters of the lifetime distribution are also taken into consideration.

Following the above concepts, the present chapter deals wit the development of a Bayesian life testing plan used for testing the lots of items or subsystems that are to be arranged in a series configuration. The and producer's risks. The concepts have been highlighted and analysed with example.

CHAPTER - 2

BAYESIAN ANALYSIS OF FIRST CONCEPITION TO LIVE BIRTH

2.0 INTRODUCTION

At the time of marriage a woman is susceptible to conception and the time elapsed before a conception is a random variable determined by fecundability, which is define as the monthly chance of a conception. It is important here to note the time of first conception after the marriage because the analysis of waiting time of first conception signifies couple's fertility at early stages of married life. This variable is widely used to study fertility characteristic of a woman, since it is independent of effect of amenorrhea period and generally, a woman does not like to use contraceptives to postpone first birth. There is little chance of recall lapse in reporting the time of first birth from the date of marriage of first birth.

Treating the first conception as a random phenomenon the probabilistic models can be developed. And this variable can be treated as discrete as well as continuous variable depending upon the situation and assumptions made for the study.

For the first time this variable was considered as a discrete variable and Gini (1924) derived the geometric distribution for the waiting time of first conception. He defined the term 'fecundability' as the monthly chance of conception for woman living in the married, fecund and exposed state.

Later the same variable was considered as a continuous variable and hence treating the same elapse from the marriage or from the beginning of the reproductive process to first conception as continuous makes mathematical treatments more convenient and easy. Singh (1964), Henry (1953) and Vincnet (1961) developed some models treating the waiting time of the first conception as continuous. The negative exponential distribution plays the role of geometric distribution for studying the waiting time of conceive after marriage. Thus, if T denotes the time of first conception then its density function, say, f(t) is given by $f(t) = \delta e^{-\delta t}; t > 0, \delta > 0$ Where δ is instantaneous fecundability.

A number of authors made modifications on the above simple distribution to study realistic situations.

In the present model of waiting time of first conception the time elapsed is defined over the range $(0, \infty)$. But in practical problems the upper limit may be considered as finite; as a woman can conceive upto

an age limit. So, there is a need of introducing a new continuous model with finite range. Keeping this in view, an attempt has been made to characterize an existing model derived by Mukeherjee Islam (1983), defined over a finite range for the purpose of life testing analysis but it suits in realistic or real life situations.

2.1 THE MODEL

A new probability distribution has been considered in this section as a continuous model, introduced by Mukhejee-Islam (1983) for the purpose of studying waiting time.

$$f(t;\theta,p) = \left(\frac{p}{\theta^p}\right)t^{p-1}; \quad p,\theta > 0; \qquad t \geq 0$$

The above model is monotonic decreasing and highly skewed to the right. The graph is J-shaped thereby showing the unimodel feature.

The same model has been modified on the range of parameter 'θ' for the specific use in the study.

$$f(t;\theta,p) = \left(\frac{p}{\theta^p}\right)t^{p-1} \quad ; \qquad \theta > 0; \quad 0 \leq t \leq \theta; \quad 0 \leq p \leq 1 \qquad \ldots\ldots. (2.1)$$

Where 'p' is instantaneous fecundability and 'θ' is considered as age limit beyond which a married woman can not conceive. Here we define the 'p' over the range (0, 1) as it has been defined in the present model as instantaneous fecundability.

The distribution function of above model will be

$$F(t) = P(T \leq t)$$

$$= \int_0^t f(t)dt$$

$$= \int_0^t \left(\frac{p}{\theta^p}\right) t^{p-1} dt$$

$$= \left(\frac{p}{\theta^p}\right) \int_0^t t^{p-1} dt$$

$$= \left(\frac{p}{\theta^p}\right) \left[\frac{t^p}{p}\right]_0^t$$

or finally

$$F(t) = \left[\frac{t}{\theta}\right]^p \qquad \text{....... (2.2)}$$

The survival function at time t, say $S(t) = P(T>t)$ is given by

$$S(t) = p[T > t]$$

$$= 1-P[T \leq t]$$

$$= 1 - F(t)$$

$$= 1 - \left[\frac{t}{\theta}\right]^p \qquad \text{.......(2.3)}$$

Also, the conception rate function, say, $w(t)$ at time t is then given by

$$w(t) = \frac{f(t;\theta, p)}{P(T \geq t)}$$

$$= \frac{\left(\frac{p}{\theta^p}\right) t^{p-1}}{1 - \left(\frac{t}{\theta}\right)^p}$$

Which finally comes out to be,

$$W(t) = \frac{pt^{p-1}}{\theta^p - t^p} \qquad \text{.......(2.4)}$$

2.2 Characterization of The Model

The important main features of the proposed model are found as follows;

2.2.1 Mean Time of First Conception

$$E(T) = \int_0^\theta t.f(t)dt$$

$$= \int_0^\theta t\left(\frac{p}{\theta^p}\right)t^{p-1}dt$$

$$= \left(\frac{p}{\theta^p}\right)\int_0^\theta t^p dt$$

$$= \frac{p}{\theta^p}\left[\frac{t^{p+1}}{p+1}\right]_0^\theta$$

i.e. $E(T) = \frac{p}{p+1}$ ....(2.5)

2.2.2 VARIANCE OF TIME OF FIRST CONCEPTION

In order to calculate V(T), firstly we calculate $E(T^2)$ as;

$$E(T^2) = \int_0^\theta t^2 f(t;\theta,p)\ dt$$

$$= \int_0^\theta t^2\left(\frac{p}{\theta^p}\right)t^{p-1}dt$$

which finally yields;

$$E(T^2) = \frac{p}{p+1}\theta^2 \qquad ....(2.6)$$

Putting the values from (2.5) and (2.6) in the following expression;

$$V(T) = E(T^2) - [E(t)]^2$$

$$= \frac{p}{p+2}\theta^2 - \left[\frac{p}{p+1}\theta\right]^2$$

$$= \frac{p}{(p+1)(p+2)}\theta^2 \qquad ....(2.7)$$

2.2.3 MAXIMUM LIKELIHOOD ESTIMATE

The likelihood function for the model (2.1) is given by

$$L(t;\theta) = p^n\theta^{-np}\prod_{i=1}^{n} t_i^{p-1}$$

$$LogL(t;\theta) = n\log p - np\log\theta + (p-1)\sum \log t_i \qquad ....(2.8)$$

Differentiating the above equation partially with respect to 'p' and equating it to zero,

$$\frac{\partial LogL(t;\theta)}{\partial\theta} = \frac{n}{p} - n\log\theta + \sum \log t_i = 0$$

The m.l.e of p is finally obtained as

$$p = \frac{n}{n\log\theta - \sum \log t_i} \qquad ....(2.9)$$

Again, differentiating the equation partially with respect to 'θ' and equating it to zero to obtain the m.l.e of θ

$$\frac{\partial LogL(t;\theta)}{\partial\theta} = \frac{np}{\theta} = 0$$

In the solution for MLE of θ the traditional method is not applicable. The MLE is obtained through order statistic technique. Since the upper limit of the model is θ, it is convincing to take $t_{(n)}$ i.e. maximum t_i as the m.l.e for the parameter θ

$$\text{i.e. } \theta = t_{(n)} = \text{Max}(t_1, t_2, \ldots\ldots, t_n) \qquad \ldots\ldots(2.10)$$

2.2.4. MEDIAN OF THE TIME OF FIRST CONCEPTION

By the definition of Median, we have

$$\int_0^{M_d} \left(\frac{p}{\theta^p}\right) t^{p-1} dt = \frac{1}{2}$$

$$\frac{p}{\theta^p}\left[\frac{t^p}{p}\right]^{Md} = \frac{1}{2}$$

$$\frac{p}{\theta^p}\left[\frac{(M_d^p)}{p}\right] = \frac{1}{2}$$

$$(M_d)^p = \frac{\theta^p}{2}$$

Hence,

$$M_d = \frac{\theta}{2^{\frac{1}{p}}} \qquad \ldots\ldots(2.11)$$

2.2.5 r^{th} MOMENT ABOUT ORIGIN OF TIME OR FIRST CONCEPTION

$$\mu_r^{\prime} = \int_0^{\theta} t^r \left(\frac{p}{\theta^p}\right) t^{p-1} dt$$

$$= \left(\frac{p}{\theta^p}\right) \int_0^{\theta} t^{p+r} dt$$

$$= \frac{p}{p+r} \theta^r \qquad \text{......(2.12)}$$

2.3 BAYESI AN ANALYSIS OF THE MODEL

In the present section an attempt is made to find out posterior distribution of the model discussed in previous section. In that particular model, there are two parameters p and θ, where p stands for fecundability and θ stands for upper age limit. Let us suppose that q varies from woman to woman according to social, economical, biological, environmental etc. factors, where p the fecundability is constant.

The three prior distributions to be considered for the purpose of the Bayesian analysis are Uniform, Inverted Gamma and the one proposed by Siddiqui et. al belongs to finite range of family of distributions while inverted Gamma belongs to infinite range of family of distributions. In the past literature it was observed that the results presented were complicated and not get very handy to solve out numerically. The

finite range priors are successfully tried in this article in order to reduce the complexity of the results.

The probability density function of the model is discussed i.e. M.I. model is as follows

$$f(t;\theta,p) = \left(p|\theta^{p}\right)t^{p-1} \qquad ......(2.13)$$

and the survival function and conception rate function of the model are

$$S(t) = 1-\left(t|\theta\right)^{p} \qquad ......(2.14)$$

and

$$w(t) = \frac{pt^{p-1}}{\theta^{p}-t^{p}} \qquad ......(2.15)$$

respectively.

2.3.1. UNIFORM PRIOR DENSITY

Now let us suppose that θ follows Uniform distribution. The probability density function of Uniform distributions; in the interval $[\alpha.\beta]$ truncated at 'a' is given by

$$g(\theta) = \frac{(a-1)(\alpha-\beta)^{a-1}}{\beta^{a-1}-\alpha^{a-1}}.\frac{1}{\theta^{a}};\ \ 0<\alpha\leq\beta<a \qquad ......(2.16)$$

Now, suppose $t_1, t_2, ..., t_n$ denotes n observation then the likelihood conditional function of θ is given by

$$L(\theta|t_1, t_2, \ldots, t_n) = \prod_{i=1}^{n} f(t_i|\theta)$$

$$= \prod_{i=1}^{n} \left(\frac{p}{\theta^p}\right) t_i^{p-1}$$

$$= p^n \theta^{-np} \prod_{i=1}^{n} t_i^{p-1}$$

$$= H.p^n \theta^{-np} \qquad \ldots\ldots(2.17)$$

where $H = \prod_{i=1}^{n} t_i^{p-1}$

The likelihood conditional function under the assumption of prior density (2.16) provides the following posterior distribution

$$G(\theta|t_1, t_2, \ldots, t_n) = \frac{H.p^n \theta^{-np} \frac{(a-1)(\alpha\beta)^{a-1}}{\beta^{a-1} - \alpha^{a-1}} \frac{1}{\theta^a}}{\int_{\alpha}^{\beta} H.p^n \theta^{-np} \frac{(a-1)(\alpha\beta)^{a-1}}{\beta^{a-1} - \alpha^{a-1}} \frac{1}{\theta^a} d\theta}$$

$$= \frac{\theta^{-np-a}}{\int_{\alpha}^{\beta} \theta^{-np-a} d\theta}$$

Which solves out to be in the following form

$$G(\theta|t_1, t_2, \ldots, t_n) = \frac{\theta^{-np-a}}{\beta^{-np-a+1} - \alpha^{-np-a+1}} (1 - a - np)$$

$$= \frac{N_1}{\theta^{np+a}}$$

where $\qquad N_1 = \dfrac{(np + a - 1)}{\dfrac{1}{\alpha^{np+a-1}} - \dfrac{1}{\beta^{np+a-1}}} \qquad \ldots\ldots(2.18)$

The Bayes estimate of the parameter θ can be obtained as

$$\theta^* = \int_{\alpha}^{\beta} \theta \frac{N_1}{\theta^{np-a}} d\theta$$

$$= \frac{N_1 \left[\frac{1}{\alpha^{np+a-2}} \frac{1}{\beta^{np+a-2}} \right]}{(np+a-2)}$$

$$= \frac{(np+a-1(\alpha\beta^{np+a-1} - \beta\alpha^{np+a-1})}{(np+a-2)(\beta^{np+a-1} - \alpha^{np+a-1})} \qquad ......(2.19)$$

The Bayes estimate of the variance of q can be obtained after obtaining the following expression.

$$E(\theta^2) = \int_{\alpha}^{\beta} \theta^2 \frac{N_1}{\theta^{np+a}} d\theta$$

$$= \frac{N_1 \left[\frac{1}{\alpha^{np+a-3}} - \frac{1}{\beta^{np+a-3}} \right]}{(np+a-3)}$$

$$= \frac{(np+a-1)}{(np+a-3)} \frac{\left[\alpha^2\beta^{np+a-1} - \beta^2_{\alpha}{}^{np+a-1}\right]}{\beta^{np+a-1} - \alpha^{np+a-1}}$$

Using these values in the following expression, variance can be calculate.

$$V^*(\theta) = E(\theta^2) - E^2(\theta)$$

$$= E(\theta^2) - (\theta^*)^2$$

$$= \frac{(np+a-1)}{(np+a-3)} \frac{\left[\alpha^2\beta^{np+a-1} - \beta^2\alpha^{np+a-1}\right]}{\beta^{np+a-1} - \alpha^{np+a-1}} - \left[\frac{(np+a-1)}{(np+a-2)} \frac{\left[\alpha\beta^{np+a-1} - \beta\alpha^{np+a-1}\right]}{\beta^{np+a-1} - \alpha^{np+a-1}} \right]^2$$

......(2.20)

The Bayes estimate of the survival function S(t) given by (2.14) will be

$$S^{*}(t)=\int_{\alpha}^{\beta}\left[1-t\left(\frac{t}{\theta}\right)^{p}\right]\frac{N_{1}}{\theta^{np+a}}d\theta$$

$$=1-t^{p}N_{1}\int_{\alpha}^{\beta}\theta^{-np-a-p}d\theta$$

$$=1-t^{p}N_{1}\frac{\left[\beta^{-p(n+1)-a+1}-\alpha^{-p(n+1)-a+1}\right]}{-p(n+1)-a+1}$$

Which has the following form: ......(2.21)

$$S^{*}(t)=1-\frac{(np+a-1)}{((n-1)p+a-1)}\frac{\left[\beta^{(n+1)p+a-1}-\alpha^{(n+1)p+a-1}\right]}{\alpha^{p}\beta^{(n+1)p+a-1}-\beta^{p}\alpha^{(n+1)p+a-1}}t^{p}$$

The Bayes' estimate of the variance of S(t) can be expressed

$$V^{*}[S(t)]=E[S^{2}(t)]-[S^{*}(t)]^{2}$$

Now,

$$E[S^{2}(t)]=\int_{\alpha}^{\beta}\left[1-\left(\frac{t}{\theta}\right)^{p}\right]^{2}\frac{N_{1}}{\theta^{np+a}}d\theta$$

$$=1+t^{2p}N_{1}\int_{\alpha}^{\beta}\theta^{-np-a-2p}d\theta-2t^{p}N_{1}\int_{\alpha}^{\beta}\theta^{-np-a-p}d\theta$$

$$=1-t^{2p}N_{1}\frac{\left[\beta^{-p(n+2)-a+1}-\alpha^{-p(n+2)-a+1}\right]}{-p(n+2)-a+1}\;-2t^{p}N_{1}\frac{\left[\beta^{-p(n+1)-a+1}-\alpha^{-p(n+1)-a+1}\right]}{-p(n+1)-a+1}$$

Which has the following form:

$$E[S^2(t)] = 1 + t^{2p}\frac{(np+a-1)}{((n-2)p+a-1)}\frac{\left[\beta^{(n+2)p+a-1} - \alpha^{(n+2)p+a-1}\right]}{\alpha^{2p}\beta^{(n+2)p+a-1} - \beta^{2p}\alpha^{(n+2)p+a-1}}$$

$$\frac{(np+a-1)}{((n-1)p+a-1)}\frac{\left[\beta^{(n+1)p+a-1} - \alpha^{(n+1)p+a-1}\right]}{\alpha^{p}\beta^{(n+1)p+a-1} - \beta^{p}\alpha^{(n+1)p+a-1}} \quad \text{......(2.22)}$$

From (2.21) and (2.22) putting the values in the expression of variance of S(t) we can obtain V*[S(t)]

- The conception rate function of the distribution given by (2.15) is

$$W(t) = \frac{pt^{p-1}}{\theta^p - t^p}$$

The Bayes estimate of conception rate w(t) will be

$$w^*(t) = \int_\alpha^\beta \left(\frac{pt}{\theta^p - t^p}\right)\cdot\frac{N_1}{\theta^{np+a}}\,d\theta = \int_\alpha^\beta \theta^{-np-a-p}\left[1-\left(\frac{t}{\theta}\right)^p\right]^{-1} d\theta$$

$$= pt^{p-1}N_1\int_\alpha^\beta \theta^{-(n+1)-a}\sum_{k=0}^{\infty}\left(\frac{t}{\theta}\right) d\theta$$

$$= pt^{p-1}N_1\sum_{k=0}^{\infty} t\frac{\left[\beta^{p(n+k+1)-a-1} - \alpha^{p(n+k+1)p+a-1}\right]}{p(n+k+1)+a-1}$$

$$= pt^{p-1}\sum_{k=0}^{\infty} t^{pk}\frac{(np+a-1)}{(n+k+1)p+a-1}\frac{\left[\beta^{p(n+k+1)+a-1} - \alpha^{p(n+k+1)p+a-1}\right]}{\left[\alpha^{(k+1)p}\beta^{p(n+k+1)+a-1} - \beta^{(k+1)} - \beta^{(k+1)p}\alpha^{p(n+k+1)p+a-1}\right]}$$

......(2.23)

Now,

$$E(w^2(t)) = p^2 t^{2(p-1)} N_1 \int_{\alpha}^{\beta} \theta^{-np-a-2p} \left[1-\left(\frac{t}{\theta}\right)^p\right]^{-2} d\theta$$

$$= p^2 t^{2(p-1)} \sum_{k=0}^{\infty} (k+1) t^{pk} \frac{(np+a-1)}{(n+k+2)p+a-1}$$

$$= \frac{\left[\beta^{p(n+k+2)+a-1} - \alpha^{p(n+k+2)p+a-1}\right]}{\left[(\alpha)^{(k+2)}(\beta)^{p(n+k+2)+a-1} - \beta^{(k+2)p}\alpha^{p(n+k+2)p+a-1}\right]} \qquad \text{......(2.24)}$$

Using the values from (2.23) and (2.24) in the following expression the Bayes estimate of variance of conception rate can be obtained.

$$V^*[w(t) = E[w^*(t)] - [w^*(t)]^2$$

The expected waiting time of first conception is obtained as

$$E^*(t) = \int_{\alpha}^{\beta} \frac{p}{p+1} . \theta \frac{N_1}{\theta^{np+a}} d\theta$$

Since

$$E(t) = \frac{p}{p+1}\theta$$

$$= \frac{p}{p+1} . N_1 \int_{\alpha}^{\beta} \theta^{-np-a+1} d\theta$$

$$= \frac{p}{p+1} \frac{(np+a-1)}{(np+a-2)} \frac{\left[\alpha\beta^{np+a-1} - \beta\alpha^{np+a-1}\right]}{\beta^{np+a-1} - \alpha^{np+a-1}} \qquad \text{......(2.25)}$$

And the variance for the same can be obtained by putting the value of

$$E^*(t^2) = \int_\alpha^\beta \frac{p}{p+2}.\theta^2 \frac{N_1}{\theta^{np+a}} d\theta$$

Since

$$E(t^2) = \frac{p}{p+2}\theta^2$$

$$= \frac{p}{p+2} N_1 \int_\alpha^\beta \theta^{-np-a+2} d\theta$$

$$E^*(t^2) = \frac{p}{(p+2)} \frac{(np+a-1)}{(np+a-3)} \frac{\left[\alpha^2\beta^{np+a-1} - \beta^2\alpha^{np+a-1}\right]}{\beta^{np+a-1} - \alpha^{np+a-1}}$$

in variance format

$$V[E^*(t)) = E^*(t^2) - [E^*(t)]^2 \qquad \text{......(2.26)}$$

2.3.2 A NEWLY DEFINED PRIOR DENSITY

In this section a new continuous distribution is being used as prior density defined by probability density function of the underlying distribution is

$$g(\theta) = \frac{h\theta^{h-1}}{q^h - h^h}\ ; \quad h \le \theta \le q;\ \ h, q > 0 \qquad \text{......(2.27)}$$

The Conditional likelihood function under the assumption of prior density (2.16) provides the following posterior distribution

$$G(q|t_1,t_2....,t_n)=\frac{H.p^n\theta^{-np}\dfrac{h\theta^{h-1}}{q^h-h^h}}{\int_\alpha^\beta H.p^n\theta^{-np}\dfrac{h\theta^{h-1}}{q^h-h^h}}$$

Which solves out to be in the following form:

$$G(\theta|t_1,t_2....,t_n)=\frac{\theta^{-np+h-1}}{\int_h^q \theta^{-np+h-1}d\theta}$$

$$=\frac{N_2}{\theta^{-np+h-1}} \qquad ......(2.28)$$

where $N_2=\dfrac{h-np}{q^{h-np}-h^{h-np}}$

The Bayes estimate of the parameter θ can be obtained as

$$\theta^*=\int_h^q \frac{\theta.N_2}{\theta^{np+h+1}}d\theta$$

$$=\frac{(h-np)}{(h-np+1)}\frac{q^{h-np+1}-h^{h-np-1}}{q^{h-np}-h^{h-np}} \qquad ......(2.29)$$

The Bayes estimate of the variance of θ can be obtained after obtaining the following expression.

$$E(\theta^2)=\int_h^q \theta^2\frac{N_2}{\theta^{np+h+1}}d\theta$$

$$=\frac{(h-np)}{(h-np+2)}\frac{q^{h-np+2}-h^{h-np+2}}{q^{h-np}-h^{h-np}}$$

Using these values in the following expression, variance can be calculated.

$$V^*(\theta) = E(\theta^2) - E^2(\theta)$$
$$= E(\theta^2) - (\theta^*)^2$$
$$= \frac{(h-np)}{(h-np+2)} \frac{q^{h-np+2} - h^{h-np+2}}{q^{h-np} - h^{h-np}} - \left[\frac{(h-np)q^{h-np+1} - h^{h-np+1}}{(h-np+1)q^{h-np} - h^{h-np}}\right]^2 \quad \text{........(2.31)}$$

The Bayes estimate of the survival function $S(t)$ given by (2.14) will be

$$S^*(t) = \int_h^q \left[1 - \left(\frac{t}{\theta}\right)^p\right] \frac{N_2}{\theta^{np+h+1}} d\theta$$
$$= 1 - t^p N_2 \int_h^q \theta^{-(n+1)p-h-1} d\theta$$
$$= 1 - t^p N_2 \frac{q^{h-(n+1)p} - h^{h-(n+1)p}}{h-(n+1)p}$$

Which has the following form:

$$S^*(t) = 1 - t^p \frac{(h-np)}{(h-(n+1)p)} \frac{q^{h(n+1)p} - h^{h(n+1)p}}{q^{h-np} - h^{h-np}} \quad \text{........(2.32)}$$

The Bayes estimate of the variance of $S(t)$ can be expressed as

$$V^*[S(t)] = E[S^2(t)] - [S^*(t)]^2$$

Now,

$$E[S^2(t)] = \int_h^q \left[1 - \left(\frac{t}{\theta}\right)^p\right]^2 \frac{N_2}{\theta^{np+h+1}} d\theta$$

$$= 1 - t^{2p} N_2 \int_h^q \theta^{-(n+2)p-h-1} d\theta$$

$$= 1 + t^{2p} N_2 \frac{q^{h-(n+2)p} - h^{h-(n+2)p}}{h-(n+2)p} - 2t^p N_2 \int_h^q \theta^{-(N+1)p-h-1} d\theta$$

Which has the following form:

$$E[S^2(t)] = 1 + t^{2p} \frac{(h-np)}{(h-(n+2)p)} \frac{q^{h(n+2)p} - h^{h(n+2)p}}{q^{h-np} - h^{h-np}} - 2t^p \frac{(h-np)}{(h-(n+1)p)} \left[\frac{q^{h(n+1)p} - h^{h(n+1)p}}{(q^{h-np} - h^{h-np})} \right]$$

......(2.33)

From (2.32) and (2.33) putting the values in the expression of variance of S(t) we can obtain V*[S(t)]

The Bayes estimate of the conception rate given by (2.15) under the assumption of prior density (2.27) will be

$$w^*(t) = \int_h^q \frac{pt^{p-1}}{\theta^p - t^p} \frac{N_2}{\theta^{np+h+1}}$$

$$= pt^{p-1} N_2 \int_h^q \theta^{-np-h-p-1} \left[1 - \left(\frac{t}{\theta} \right)^p \right] d\theta$$

$$= pt^{p-1} N_2 \int_h^q \theta^{-p(n+1)-h-1} \sum_{k=0}^{\infty} \left(\frac{t}{\theta} \right)^{pk} d\theta$$

Solving the above expression we get:

$$= pt^{p-1} \sum_{k=0}^{\infty} t^{pk} \frac{(h-np)}{(h-(n+k+1)p)} \frac{q^{h-(n+k+1)p} - h^{h-(n+k+1)p}}{(q^{h-np} - h^{h-np})} \qquad ......(2.34)$$

Now, to obtain Bayes estimate of the variance of the same

$$E[w^2(t)] = \int_h^q \left[\frac{pt^{p-1}}{q^p - t^p} \right] \frac{N_2}{\theta^{np+h+1}} d\theta$$

$$= p^2 t^{2(p-1)} N_2 \int_h^q N_2 \theta^{-np-h-2p-1} \left[1 - \left(\frac{t}{\theta} \right)^p \right]^{-2} d\theta$$

$$= p^2 t^{2(p-1)} N_2 \int_h^q \theta^{-p(n+2)-h-1} \sum_{k=0}^{\infty} (k+1)\left(\frac{t}{\theta}\right)^{pk} d\theta$$

Solving the above expression we get;

$$= p^2 t^{2(p-1)} \sum_{k=0}^{\infty} (k+1) t^p \frac{(h-np)}{(h-(n+k+2)p)} \frac{q^{h-(n+k+2)p} - h^{h-(n+k+2)p}}{q^{h-np} - h^{h-np}} \quad \text{......(2.35)}$$

Putting the values from (2.34) and (2.35) in following expression

$V^*(w(t)) = E(w^2(t)] - [w^*(t)]^2$

The expected waiting time of first conception under the assumption of prior density (2.27) is obtained as :

$$E^*(t) = \int_{ha}^{q} \frac{p}{p+1} . \theta . \frac{N_2}{\theta^{np+h+1}} d\theta \qquad \left(\because E(t) = \frac{p}{p+1}\theta\right)$$

$$= \frac{p}{p+1} N_2 \int_h^q . \theta^{-np-h} d\theta$$

$$= \frac{p}{p+1}\left[\frac{h-np}{(h-np+1)} \frac{q^{h-np+1} - h^{h-np+1}}{q^{h-np} - h^{h-np}}\right] \quad \text{......(2.36)}$$

And

$$E^*(t^2) = \int_{ha}^{q} \frac{p}{p+2} . \theta^2 . \frac{N_2}{\theta^{np+h+1}} d\theta \qquad \left(\because E(t) = \frac{p}{p+1}\theta^2\right)$$

$$= \frac{p}{p+2} N_2 \int_h^q . \theta^{-np-h-1} d\theta$$

$$= \frac{p}{p+2}\left[\frac{h-np}{h-np+2} \frac{q^{h-np-2} - h^{h-np+2}}{q^{h-np} - h^{h-np}}\right] \quad \text{......(2.37)}$$

Putting the values from (2.36) and (2.37) in following expression, we can get the expression for the Bayes estimate of variance of the expected waiting time of first conception.

$$V^*(E) = E^*(t^2) - [E^*(t)]^2$$

2.3.3. INVERTED GAMMA PRIOR DENSITY

The probability density function of inverted gamma distribution is;

$$g(\theta) = \left(\frac{v}{\theta}\right)^{\mu+1} \frac{e^{-\left(\frac{v}{\theta}\right)}}{\Gamma(\mu+1)} \qquad v,\mu,\theta \geq 0 \qquad \text{......(2.38)}$$

The applicability of this model in the Bayesian analysis is already being discussed by Chhikara and Folks (1971). The distribution reduces to exponential density for $\mu = 0$

The likelihood conditional under the assumption of prior density (2.3.8) provides the following posterior distribution.

$$G(\theta|t_1, t_2, \ldots\ldots, t_n) = \frac{H.p^n\theta^{-np}\left(\frac{v}{\theta}\right)^{\mu+1} \frac{e^{-\left(\frac{v}{\theta}\right)}}{\Gamma(\mu+1)}}{\int_0^\infty H.p^n\theta^{-np}\left(\frac{v}{\theta}\right)^{d+1} \frac{e^{\left(\frac{v}{\theta}\right)}}{\Gamma(d+1)} d\theta}$$

which solves out to be in the following form:

$$G(\theta|t_1,t_2.....,t_n)=\frac{\theta^{-np-\mu-1}e^{-\left(\frac{v}{\theta}\right)}}{\int_0^\infty \theta^{-np-\mu-1}e^{-\left(\frac{v}{\theta}\right)}d\theta}$$

$$G(\theta|t_1,t_2.....,t_n)=(\theta)^{-np-\mu-1}e^{-\left(\frac{v}{\theta}\right)}N_3 \qquad ......(2.39)$$

where $N_3=\dfrac{\gamma}{\Gamma(np+\mu)}$.

The Bayes estimate of the parameter q can be obtained as;

$$0^*=\int_0^\infty \theta.\theta^{-np-\mu-1}e^{\left(\frac{v}{\theta}\right)}N_3 d\theta$$

$$=N_3\int_0^\infty \theta^{-np-\mu-1}.e^{\left(\frac{v}{\theta}\right)}d\theta$$

which solves out to be as;

$$0^*=\frac{v}{(np+\mu-1)} \qquad ......(2.40)$$

The Bayes estimate of the variance of θ is obtained after obtaining the following expression.

$$E(\theta^2)=\int_0^\infty \theta^2.\theta^{-np-\mu-1}e^{-\left(\frac{v}{\theta}\right)}N_3 d\theta$$

$$=N_3\int_0^\infty \theta^{-np-\mu-1}e^{-\left(\frac{v}{\theta}\right)}d\theta$$

which solves out to be as;

$$E(\theta^2)=\frac{v^2}{(np+\mu-2)(np+\mu-1)} \qquad ......(2.41)$$

Putting these values in the expression variance can be calculated as

$$V^*(\theta) = E(\theta^2) - (\theta^*)^2$$

$$= \frac{v^2}{(np+\mu-2)(np+\mu-1)}\left[\frac{v}{(np+\mu-1)}\right]^2$$

$$V^* = \frac{v^2}{(np+\mu-2)(np+\mu-1)^2} \qquad ......(2.42)$$

The Bayes estimate of survival function under the assumption of prior density (2.25) is obtained as

$$S^*(t) = \int_0^\infty \left[1-\left(\frac{t}{\theta}\right)^p\right]\theta^{-np-\mu-1}e^{-\left(\frac{v}{\theta}\right)}N_3 d\theta$$

$$= 1 - t^p N_3 \int_0^\infty \theta^{-(n+1)p-\mu-1}e^{-\left(\frac{v}{\theta}\right)}d\theta$$

$$= 1 - t^p N_3 \frac{\Gamma[(n+1)p+\mu]}{v^{(n+1)p-\mu}}$$

Which has the following form:

$$S^*(t) = 1 - t^p v^p \frac{\Gamma[(n+1)p+\mu]}{\Gamma(np+\mu)} \qquad ......(2.43)$$

The Bayes estimate of variance of survival function can be expressed as

$$V^*[S(t)] = E[S(t)] - [S^*(t)]^2$$

where

$$E[S^2(t)] = \int_0^\infty \left[1-\left(\frac{t}{\theta}\right)^p \theta\right]^2 \theta^{-np-\mu-1}e^{-\left(\frac{v}{\theta}\right)}N_3 d\theta$$

$$= 1 + t^2 N_3 \int_0^\infty \theta^{-(n+2)p-\mu-1} e^{-\left(\frac{v}{\theta}\right)} d\theta$$

$$-2t^2 N_3 \int_0^\infty \theta^{-(n+2)p-\mu-1} e^{\left(\frac{v}{\theta}\right)} d\theta$$

$$= 1 + t^{2p} v^{2p} \frac{\Gamma[(n+2)p+\mu]}{\Gamma(np+\mu)} - 2t^p . v^p \frac{\Gamma[(n+1)p+d]}{\Gamma(np+d)} \qquad ......(2.44)$$

The Bayes estimate of conception rate given by (2.14) under the assumption of the prior density (2.27) is obtained as

$$w^*(t) = \int_0^\infty \frac{pt^{p-1}}{\theta^p - t^p} \theta^{-np-p-1} e^{-\left(\frac{v}{\theta}\right)} N_3 d\theta$$

$$= pt^{p-1} N_3 \int_0^\infty \theta^{-p(n+2)-\mu-1} e\left(\frac{v}{\theta}\right) \sum_{k=0}^\infty \left(\frac{t}{\theta}\right)^{pk} d\theta$$

$$= pt^{p-1} N_3 \sum_{k=0}^\infty t^{pk} \int_0^\infty \theta^{-p(n+k+1)-\mu-1} e^{-\left(\frac{v}{\theta}\right)} d\theta$$

Solving the above expression we get;

$$w^*(t) = pt^{p-1} N_3 \sum_{k=0}^\infty t^{pk} v^{(k+1)p} \frac{\Gamma[p(n+k+1)+\mu]}{\Gamma[(np+5)} \qquad .....(2.45)$$

and also,

$$E^*[w^2(t) = \int_0^\infty \left[\frac{pt^{p-1}}{\theta^p - t^p}\right] \theta^{-np-\mu-1} e^{-\left(\frac{v}{\theta}\right)} N_3 d\theta$$

$$= p^2 t^{2(p-1)} N_3 \int_0^\infty \theta^{-p(n+2)-d-1} e^{-\left(\frac{v}{\theta}\right)} \sum_{k=0}^\infty (k+1) \left(\frac{t}{\theta}\right)^{pk} d\theta$$

Solving the above expression we get;

$$E^*[w^2(t)] = p^2 t^{2(p-1)} \sum_{k=0}^\infty (k+1)(t^{pk} v^{(k+2)p} \frac{\Gamma[p(n+k+2)+p]}{\Gamma[(np+6)} \qquad ....(2.46)$$

Now, the Bayes estimate of the variance of conception rate can be obtained by putting values from (2.45) and (2.46) in the following expression.

$$V^*[w(t)] = E(w^2(t)] - [w^*(t)]^2$$

The expected waiting time off first conception under the assumption of prior density (2.25) is obtained as:

$$E^*(t) = \int_0^\infty \frac{p}{p+1}.\theta.\theta^{-np-\mu-1} e^{-\left(\frac{v}{\theta}\right)} N_3 d\theta \qquad \left(\because E(t) = \frac{p}{p+1}.\theta\right)$$

$$= \frac{p}{p+1}\int_0^\infty \theta^{-np-\mu} e^{-\left(\frac{v}{\theta}\right)} N_3 d\theta$$

$$= \frac{p}{p+1}\frac{v}{np+\mu-1} \qquad .....(2.47)$$

And

$$E^*(t^2) = \int_0^\infty \frac{p}{p+2}.\theta^2.\theta^{-np-\mu-1} e^{-\left(\frac{v}{\theta}\right)} N_3 d\theta \qquad \left(\because E(t^2) = \frac{p}{p+2}.\theta^2\right)$$

$$= \frac{p}{p+2}\int_0^\infty \theta^{-np-\mu+1} e^{-\left(\frac{v}{\theta}\right)} N_3 d\theta$$

$$= \frac{p}{p+2}\frac{v^2}{(np+\mu-1)(np+\mu-2)} \qquad .....(2.48)$$

Now putting the values of (2.47) and (2.48) in the following expression, we can obtain the variance for the same as;

$$V^{*}(E) = E^{*}(t^{2}) - [E^{*}(t)]^{2}$$

$$= \frac{p}{p=2}\frac{v^{2}}{(np+\mu-1)(np+\mu-2)} - \left[\frac{p}{p+1} - \frac{v}{(np+\mu=1)}\right]^{2}$$

$$= \frac{pv^{2}}{(np+\mu-1)}\left[\frac{1}{(p+2)(np+\mu-1)} - \frac{p}{[(p+1)(np+\mu-1)]^{2}}\right] \qquad .....(2.49)$$

CHAPTER - 3

STUDY OF BURR TYPE III DISTRIBUTION USING TYPE I CENSORED DATA

3.1 Introduction

Generally there are two type of censoring Type I and Type II censoring. In Type I censoring, the number of failure is random whereas the length of the experiment time T is fixed. In Type II censoring, the number of failures r, is fixed while the length of experiment time T is random variable.

This study considers Burr Type III distribution, introduced by Burr (1942) as a versatile unimodel distribution. The Burr Type III family is much richer than Burr Type XII as is shown by the moment ratio diagrams constructed by Rodriguez (1977, 1982) Burr Type III distribution includes exponential, Weibull and Log-logistic distribution as special cases. The curve shape characteristics for Pearson Type I (Beta), II, V, VII, IX and XII families (see Rodriguez (1977, 1982) is also present in Burr Type III distribution. As the model based on this

distribution is very rich and simplest functionally therefore it is very useful and most attractive for statistical modelling. Wingo (1993) has described method for fitting the Burr type XII distribution to complete life test data by maximum likelihood method. Recently, Wingo (1993) consider Burr Type XII distribution and discussed Maximum Likelihood method for fitting the Burr Type XII distribution to progressively censored life test data.

The present investigation extends the Wingo (1993) work by considering Burr Type III distribution, which is much richer than Burr XII. It also extends the work of Wingo (1983) by developing mathematical and computational methodology for fitting the Burr Type II distribution to type I progressively censored life test data. Mathematical expressions are obtained for finding the asymptotic variances and covarience of the MLEs of the parameter of the distribution. The existence and uniqueness of the MLEs for arbitrary sample data are also discussed.

3.2 Models and Assumption

Let $\{X_i > 0, i = 1,2......,n\}$ be the independent lifetimes random variable having a two-parameter Burr Type III distribution with the cumulative distribution function $F(x) = [1 + x^{-c}]^{-k}; \quad x, c, k > 0$ .....(3.1)

Where c and k are the shape parameters. Thus the probability density function is $\frac{d}{dx}F(x) = -k(1+x^{-c})^{-(k+1)}(-cx^{-(c+1)})$

$$f(x) = \frac{kcx^{-(c+1)}}{(1+x^{-c})^{k+1}} \qquad \text{......(3.2)}$$

Putting k=1, Burr III reduced to log-logistic with probability density function is given by $f(x) = \frac{cx^{-(c+1)}}{(1+x^{-c})^2}$ ......(3.3)

The Hazard rate function of Burr III distribution becomes

$$h(t) = \frac{f(t)}{(1-F(t)} \qquad \text{......(3.4)}$$

$$= \frac{\frac{kcx^{-(c+1)}}{(1+x^{-c})^{k+1}}}{1-(1+x^{-c})^{-k}}$$

$$= \frac{kcx^{-(c+1)}}{(1+x^{-c})^{k+1}\{1-(1+x^{-c})^{-k}}$$

$$h(t) = \frac{kcx^{-(c+1)}}{(1+x^{-c})^{k+1}-(1+x^{-c})} \qquad \text{...... (3.5)}$$

and the quartile function of the Burr type III distribution is

$$p = \int_0^{Q(p)} f(x)dx$$

$$F = (Q(p)) = p$$

$$[1+(Q(p)^{-c}]^{-k} = p$$

$$1+(Q(p)^{-c} = p^{-1/k}$$

$$(Q(p)^{-c} = p^{-1/k} - 1$$

$$(Q(p))^{-c} = \frac{1}{p^{1/k}} - 1$$

$$(Q(p))^{-c} = \frac{1-p^{1/k}}{p^{1/k}}$$

$$Q(p) = \frac{p^{1/kc}}{(1-p^{1/k})^{1/c}} - 1$$

Suppose that the total number of items (N) are placed life testing experiment and let n be the number of known exact failure. Further suppose that the censoring occurs j = 1,2,3.....,m. If, at j^{-th} stage of censored the randomly selected items r_j, are removed or censored from the test. Thus we can write.

$$N = n + \sum_{j=1}^{m} r_j \qquad \ldots\ldots (3.6)$$

and the likelihood function for m-stage Type I progressively censoring is given by

$$L^{!} = \lambda \prod_{n=1}^{n} f(x_1) \prod_{j=1}^{m} [1 - F(T_j)]^{r_j} \qquad \ldots\ldots (3.7)$$

for some constant λ.

3.3. Maximum Likelihood Estimation

The log – likelihood function (LLE) based on the I-state Type-I progressively censored sample, can be written as

$$L = In \;\; L^{!} = n\log k + n\log c - (c+1).\sum_{i=1}^{n}\log x_i - (k+1)\sum_{i=1}^{n}\log(1+x^{-c}) + \sum_{j=1}^{n} r_j \log[1-(1+T_j^{-c})-k] \qquad \ldots\ldots (3.8)$$

Differentiating partially the above equation with respect to c and k respectively, we get the following likelihood equations

$$\frac{\partial L}{\partial c}=\frac{n}{c}-\sum_{i=1}^{n}\log x_i+(k+1)\sum_{i=1}^{n}\frac{x_i^{-c}\text{InXi}}{(1+\text{Xi}^{-c})}-k\sum_{j=1}^{m}r_j\frac{(1+T_j^{-c})^{-(k+1)}T_j^{-c}\log T_j}{[1-(1+T_j^{-c})^{-k}]} \quad \text{..... (3.9)}$$

$$\frac{\partial L}{\partial k}=\frac{n}{k}-\sum_{i=1}^{n}\log(1+x^{-c})-\sum_{j=1}^{n}\frac{kr_j(1+T_j^{-c})^{-(k+1)}}{[1-(1+T_j^{-c})^{-k}]}=0 \quad \text{..... (3.10)}$$

The MLE's $\hat{c}$ and $\hat{k}$ of c and k respectively are the solutions of the above equations (3.9) and (3.10). These equation can not be solved explicitly. But they may be solved by using numerical method in order to determine the values of c and k.

3.4 Asymptotic Variances and Covariance's of MLEs

The observed information matrix F, is obtained by taking negative of second partial derivative of log likelihood function L with restart to c and k. Thus

$$F=\begin{bmatrix}\sum_{i=1}^{n}\left[\frac{-\partial^2\log L_i}{\partial c^2}\right] & \sum_{i=1}^{n}\left[\frac{-\delta^2\text{InL}_i}{\delta c\delta k}\right]\\ \sum_{i=1}^{n}\left[\frac{-\partial^2\log L_i}{\partial k\partial c}\right] & \sum_{i=1}^{n}\left[\frac{-\partial^2\text{InL}_i}{\partial k^2}\right]\end{bmatrix}$$

$$F=n\begin{bmatrix}f_{00}, & f_{01}\\ f_{10} & f_{11}\end{bmatrix} \quad \text{...... (3.11)}$$

Where $fgh=-\frac{\partial^2 L}{\partial\theta g\ \partial\theta n}$; $g,h=0,1$

3.5 Existence and Uniqueness of MLE'S

Case I : Suppose k is known and fixed, we have to show that the estimator c exists and is unique. For uniqueness, it s to show that the function defined in (3.8) is strictly concave in c. Since $\frac{\partial^2 L}{\partial c^2}$ is negative, it implies that the function is strictly concave in. Hence c is. For existence of c, we have to show that the equation $\frac{\partial^2 L}{\partial c^2} = 0$ (ie eq 3.9) has a single zero, finite zero on the positive real line, provided that $X_i \neq 1$ and / or $Xj \neq 1$ for some i=1, 2....., n and j = 1, 2 ....., m.

Let us define eq. (3.9) as a function of c only and index sets as

$$\phi(c) = \frac{N}{C} k \sum_{j=1}^{n} InX_i - (k+1) \sum_{j=1}^{n} \frac{X_i^{\,c} In}{1 + X_i^{\,-c}} - k \sum_{j=1}^{1} \frac{r_i T^c InT_i}{I + X_i^{\,-c}}$$

$$+ k \sum_{j=1}^{1} \frac{r_j (l + T_j^c)^{k-1} T_j^c - T^{ck}) \ln T_2}{(1 + T_j^c) - T_j^{ck}}$$

and

$M = \{i : X_i < 1, i = 1, 2, \ldots\ldots, n\}$

$P = \{i : X_i < 1, i = 1, 2, \ldots\ldots, n\}$

$Q = \{j : T_j < 1, j = 1, 2, \ldots\ldots, m\}$

$R = \{j : T_j < 1, j = 1, 2, \ldots\ldots, m\}$ .... (3.16)

As the observed asymptotic variance-covariance matrix of estimators $\hat{c}$ and $\hat{k}$ is the inverse of the observed information matrix F, therefore

$$V = F^{-1}\begin{bmatrix} \operatorname{var}(\hat{c}) & \operatorname{cov}(\hat{c},\hat{k}) \\ \operatorname{cov}(\hat{k},\hat{c}) & \operatorname{var}(\hat{k}) \end{bmatrix}$$

$$n = n^{-1}\left[f_{00}f_{11} - f_{01}^{2}\right]\begin{bmatrix} f_{11} & f_{01} \\ f_{10} & f_{00} \end{bmatrix}^{-1}$$

After some algebraic simplification, we obtain

$$\frac{\delta^2 L}{\delta c^2} = -\frac{n}{c^2} + (k+1)\sum_{i=1}^{n}\left[\frac{-(1+x^{-c})x^{-c}(\ln x)^2 + (x^{-c}\ln x)^2}{(l+x^{-c})^2}\right]$$

$$-\sum_{j=1}^{l} r_j k \left[\frac{\begin{array}{c}\{1-(1+T_j^{-c})^{-k}\}\{-cT_j^{-c}(1+T_j^{-c})^{-(k+1)}(\ln T_j)^2 + (k+1)(1=T_j^{-c})^{-(k+2)}(T_j^{-c}\ln T_j)^2\} \\ -\{(1+T_j^{-c})^{-(k+1)}T_j^{-c}\ln T_j\}\{k(1+T_j^{-c})^{-(k+1)}T_j^{-c}\ln T_j\}\end{array}}{\{1-(1+T_j^{-c})^{-k}\}^2}\right]$$

$$\frac{\delta^2 L}{\delta k^2} = -\frac{n}{k^2} - \sum_{j=1}^{n} kr_j\left[\frac{-(k+1)(1+T_j^{-c})^{-(k+2)} + (k+1)(1+T_j^{-c})^{-2(k+1)} - k(1+T_j^{-c})^{-2(k+1)}}{\left[1-(1+T_j^{-c})^{-k}\right]^2}\right]$$

$$\frac{\delta^2 L}{\delta k^2} = -\frac{n}{k^2} - \sum_{j=1}^{n} kr_j\left[\frac{-(k+1)(1+T_j^{-c})^{-(k+2)} + (1+T_j^{-c})^{-2(k+1)}}{\left[1-(1+T_j^{-c})^{-k}\right]^2}\right]$$

After some algebraic simplification and direct calculation under the above non-empty index sets, as $c \to 0$ and as $c \to \infty$, equation (3.15) yields respectively $\phi(0) = \infty$

$$\phi(\infty) = k \sum_{i=M} \log x_i - \sum_{i=P} \log x_i - k \sum_{i=R} r_j \mathrm{In} T_j \qquad \ldots\ldots.. (3.17)$$

The right hand side of (3.17) will remain negative, as long as either of the index sets M, P, Q or R. Therefore under these conditions on the sample data we may say that $\phi(0) < \infty$ and $\phi(\infty) < 0$, there exist at least one positive real line.

Differentiation and straight forward algebraic simplification show that $\frac{d\phi(c)}{dc} < 0$ (i.e. $\phi(c)$ is monotonic decreasing function in c), implying this root $\hat{c}$ is unique. Therefore we conclude that $\phi(c)=0$ has a single finite zero on the positive real line. Hence the MLE's is always exits and is unique for known and fixed k.

Case II : Suppose c is known and fixed, the function in (3.8) is strictly concave in k, because eq. (3.14) is negative. It follows from the strict concavity of (3.8) that k is unique now we have to show that k always exit possesses a single, finite zero on the real line provided that $x_i \neq 1$ and / or $T_j \neq 1$ for some i = 1, 2, ..., n and j = 1, 2, ..., m.

We define

$$\Psi(k) = \frac{n}{k} + c\sum_{i=1}^{n} \log x_i - \sum_{i=1}^{n} \log(1 + x_i^{-c}) - \sum_{j=1}^{t} r_j(1 + T_j^c) + \sum_{j=1}^{t} \frac{r_j[(1+T_j^{-c})^k \log(1+T_j^c) - cT_j^{ck} \mathrm{In} T_j]}{(1+T_j^c)^k - T_j^{ck}}$$

After some algebraic manipulation and direct calculation of eq. (3.18) over non empty index set defined in eq. (3.16) as $k \to 0$ and $k \to \infty$ we get $\Psi(\infty) = -c\sum_{i=1}^{n}\log(Xi) - \sum_{i=1}^{n}\log(1 + xi^{-c}) - \sum_{j \in Q} r_j(1 + T_j^c)$ we see that right-hand side of the eq.(3.10) is negative if and only if index set Q or R is non-empty (i.e. $\phi(\infty) < 0$). thus $\phi(0) < \infty$ and $\phi(\infty) < 0$, implies that there exist at least one real root of $\Psi(k) = \frac{\delta L}{\delta k} = 0$. one can easily show that of $\frac{\partial \Psi(k)}{\partial k} < 0$. i.e. $\psi(k)$ is monotonic decreasing in k.

It follows that this root $\hat{k}$ is unique Therefore we conclude that the eq. $\Psi(k) = 0$ has a single, finite zero on the positive real line Hence when c is known and fixed, the MLE k always exists and is unique.

3.6 Conclusion

This chapter develops mathematical methodology for estimating the parameters involved in Burr Type III distribution from Type I progressively censored life test data by maximum likelihhod method Asymptotic variances and covarience's of MLEs are obtain. We observe that the MLEs of the parameter exist, unique and finite for arbitrary sample data. We also observed that the result of this chapter be applied to type I progressively censored sample data arising in a life test experiment in clinical trials and fields where Burr Type III distribution is used.

CHAPTER - 4

A STUDY OF GUMBEL'S BIVARIATE EXPONENTIAL DISTRIBUTION FOR LIFE TESTING ANALYSIS

4.1. Introduction

The most widely used and accepted model for life testing is the exponential model. The density function is given by:

$$f(\theta, t) = \theta \exp(-\theta . t); \qquad \theta, t \geq 0$$

The two important properties of this model viz. Forgetfulness and constant hazard rate make this model even more popular in life testing analysis.

The constant hazard rate means that the exponential distribution is an appropriate model for the life time of an item when there is no ageing or wear out. It is mostly applicable to electronic components since they fail due to chance and not due to wear out or ageing. This model is equally popular in the similar branches of studies like fertility and survival analysis.

The utility of it, leads a number of workers to develop the bivariate (multivariate) extension of this distribution. There are a number of bivariate exponential distributions.

In the present chapter an attempt has been made to characterize the concomitants of the order statistics of Gumbel's exponential bivariate distribution.

Let (x_i, y_i), i = 1, 2, ..., n be n pairs of independent random variables from Gumbel's exponential bivariate distribution (which may be consider as a particular case of Gumbel's Weibull bivariate distribution) with the distribution function.

$$F(x,y) = 1 - \exp[-ax] - \exp[by] + \exp[-ax - by - cxy];$$

$$(x, y \geq 0), (a, b > 0), (0 \leq c \leq ab) \qquad \text{...... (4.1)}$$

and the corresponding probability density function can be obtained as;

$$f(x,y) = \frac{d}{dy}\left\{\frac{d}{dx}[F(x,y)]\right\}$$

$$= \frac{d}{dy}\{a.\exp(-ax) - (a+cy).\exp[-ax - by - cxy]\}$$

$$= \{(a+cy)(b+cx)\exp[-ax - by - cxy] - c\exp[-ax - by - cxy]\}$$

which finally yields;

$$f(x,y) = [b+cx)(a+cy) - c]\exp(-ax - by - cxy);$$

$$(x, y \geq 0), (a, b > 0), (0 \leq c \leq ab) \qquad \text{...... (4.2)}$$

4.2 Marginal Probability Density Functions

In the present section the marginal probability density functions and distribution functions of X and Y are obtained.

The marginal probability density function of X can be obtained as;

$$g(x)=\int_{-\infty}^{\infty} f(x,y)dy=\int_{0}^{\infty} f(x,y)dy$$

$$=\int_{0}^{\infty}[(b+cx)(a+cy)-c]\exp(-ax-by-cyx)dy$$

$$=e^{-ax}\int_{0}^{\infty}[ab+bcy+acx+c^{2}xy-c]\exp[-y(b+cx)]dy$$

$$=e^{-ax}\left[\int_{0}^{\infty} cy^{2-1}(b+cx)e^{-y((b+cx)}dy+(ab+acx-c)\int_{0}^{\infty} e^{-y(b+cx}dy\right]$$

Since $\int_{0}^{\infty}\exp(-pt)t^{v}dt=p^{-(v+1)}\Gamma(v+1)$

$$=e^{-ax}\left\{\frac{c(b+cx)}{(b+cx)^{2}}+\frac{(ab+acx-c)}{(b+cx)}\right\}$$

$$=e^{-ax}\left\{\frac{c+ab-c+acx}{(b+cx)}\right\}$$

$$=e^{-ax}\left\{\frac{a(b+cx)}{(b+cx)}\right\}$$

$$=ae^{-ax}$$

which finally yields

$$g(x)=a\exp(-ax)\quad ;\ x>0 \qquad \text{...... (4.3)}$$

And the marginal distribution function of X is;

$$G(x) = \int_0^x g(x)dx$$

$$= \int_0^x a \ \exp(-ax)dx$$

$$G(x) = 1 - \exp(-ax) \qquad ; x > 0 \qquad \text{...... (4.4)}$$

Similarly, the marginal probability density function of Y will be;

$$h(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^{\infty} f(x,y)dx$$

$$= \int_0^{\infty} [(b+cx)(a+cy) - c]\exp(-ax - by - cxy)dx$$

$$= e^{-by} \int_0^{\infty} [ab + bcy + acx + c^2xy - c]\exp[-x(b+cx)]dx$$

$$= e^{-by} \int_0^{\infty} [x(ac + c^2y) + (ab + bcy - c)]\exp[-x(a+cy)]dx$$

$$= e^{-by} \left\{ \frac{b(a+cy)}{(a+cy)} \right\}$$

which finnaly yields

$$h(y) = b \ \exp(-by) \qquad ; y > 0 \qquad \text{....... (4.5)}$$

The mean of the random variable Y having the probability density function as defined in (4.5) will be;

$$E(Y) = \int_0^{\infty} y.h(y)dy$$

$$= \int_0^\infty yb\exp(-by)dy$$

$$G(Y) = \frac{1}{b} \qquad ....(4.6)$$

Also;

$$E(Y^2) = \int_0^\infty y^2 h(y)dy$$

$$= \int_0^\infty y^2 b\exp(-by)dy$$

$$E(Y^2) = \frac{2}{b^2} \qquad ....(4.7)$$

And the marginal distribution function of Y is;

$$H(y) = \int_0^y h(y)dy$$

$$= \int_0^y b\exp(-by)dy$$

$$H(y) = 1-\exp(-by) \quad ; y > 0 \qquad ....(4.8)$$

4.3 Conditional Probability Density Functions

Now, in the present section the conditional probability density functions of Y for given X and of X for given Y are obtained.

The conditional probability density function of Y for given X can be obtained as follow;

$$f(y|x) = \frac{f(x,y)}{g(x)}$$

$$f(y|x) = \frac{[(b+cx)(a+cy)-c]\exp[-y(b+cx)]}{a} \qquad \text{.......(4.9)}$$

Similarly, the conditional probability density function of X for given Y will as follows;

$$f(x|y) = \frac{f(x,y)}{h(x)}$$

$$= \frac{[(b+cx)(a+cy)-c]\exp(-ax-by-cxy)}{b\exp(-by)}$$

$$f(x|y) = \frac{[(b+cx)(a+cy)-c]\exp(-x(a+cy)]}{b} \qquad \text{......(4.10)}$$

4.4 Probability Density Function of Order Statistics From Gumbel's Exponential Bivariate Distribution

In the present section, the probability density function of the r^{th} ordered statistics and the joint probability density function of r^{th} and s^{th} ordered statistics from Gumbel's exponential bivariate distribution are obtained.

For Gumbel's exponential bivariate distribution the probability density function of the r^{th} order statistics $X_{r;n}$ is;

$$f_{rn}(x) = C_{rn}[F(x)]^{r-1}[1-F(x)]^{n-r} f(x); \quad 0 < x < \infty$$

Where $C_{rn} = \frac{n!}{(r-1)!(n-r)!}$

$$= f_{r:n} = C_{rn}[1-\exp(-ax)]^{r-1}\exp[-a(n-r+1)x] \quad ....(4.11)$$

In particular for r = 1, i.e. the probability density function of the first order statistics is;

$$f_{r:n}(x) = na\exp(-nax) \quad ......(4.12)$$

For the distribution with probability density function (4, 3), the joint distribution of two order statistics r^{th} and s^{th} is as follows;

$$f_{r,s:n}(x_1,x_2) = C_{r,s:n}[F(x_1)]^{r-1}[F(x_2)-F(x_1)]^{s-r-l}[1-F(x_2)]^{n-s}f(x_1)f(x_2)$$

where $C_{r,s:n} = \frac{n!}{(r-1)(s-r-1)!(n-s)!}$

$$f_{r,s:n}(x_1,x_2) = C_{r,s:n}[1-\exp(-ax_1)]^{r-1}$$

$$\{[1-\exp(-ax_2)]-[1-\exp(-ax_1)]\}^{s-r-1}$$

$$[\exp(-ax_2)]^{n-s}[a\exp(-ax_1)][a\exp(-ax_2)]$$

$$= C_{r,s:n}[1-\exp(-ax_1)]^{r-1}\{\exp(-ax_1)-\exp(-ax_2)\}^{s-r-1}$$

$$\cdot[\exp(-ax_2)]^{n-s}a^2\exp[-a(x_1+x_2)]$$

$$= C_{r,s:n}a^2\sum_{i=0}^{r-1}\sum_{j-0}^{s-r-1}(-1)^{(i+j)}\begin{bmatrix} r-1 \\ i \end{bmatrix}\begin{bmatrix} s-r-1 \\ j \end{bmatrix}\exp(-ia+sa-ra-a-ja)x_1)]$$

$$[\exp(-jax_2)]\exp[-(na-sa)x_2]\exp[-a(x_1+x_2)]$$

which finally yields

$$f_{r,s:n}(x_1x_2) = C_{r,s:n}a^2\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{(i+j)}\begin{bmatrix} r-1 \\ i \end{bmatrix}\begin{bmatrix} s-r-1 \\ j \end{bmatrix}$$

$$\exp(-(ia+sa-ra-ja)x_1)]\exp(-na-sa+ja+a)x_2] \quad ......(4.13)$$

$$= C_{r,s:n}a^2\sum_{i-0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{i}\binom{s-r-1}{j}\exp(-(i+s-r-j)ax_1)\exp-(n-s+j+1)ax_2$$

4.5 Probability Density Function of Concomitants

Now, the probability density function of the first order concomitant (i.e. r=1) of the order statistics is

$$g_{(1:n)}(y) = \int_0^\infty f(y|x) f_{1:n}(x)dx$$

$$= \int_0^\infty \frac{[(b+cx)(a+cy)-c]\exp[-y(b+cx)]}{a} na\exp(-nax)dx$$

$$= n\int_0^\infty [(b+cx)(a+cy)-c]\exp[-y(b+cx)]\exp(-nax)dx$$

$$= n\exp(-by)\int_0^\infty [cx(a+cy)+ab-c+bcy]\exp[-x(na+cy)]dx$$

Now using the following relation,

$$\int_0^\infty \exp(-pt)p^{v}dt = p^{-(v+1)}\Gamma(v+1) \qquad \mathrm{Re}(v) \geq -1, \mathrm{Re}(p) \geq 0$$

$$= n\exp(-by)\left\{\frac{c(a+cy)}{(cy+na)^2} + \frac{(ab-c+bcy)}{(cy+na)}\right\}$$

After simplification,

$$g_{(1:n)}(y) = \exp(-by)\left\{nb - \frac{abn(n-1)}{c\left(y+\frac{na}{c}\right)} - \frac{an(n-1)}{\left(y+\frac{na}{c}\right)^2}\right\} \quad ; y \geq 0 \qquad \text{......(4.14)}$$

Similarly, $g_{(1:n)}(x)$ can be obtained as;

$$g_{(1:n)}(x) = \exp(-by)\left\{ nb - \frac{abn(n-1)}{c\left(y+\frac{nb}{c}\right)} - \frac{bn(n-1)}{c\left(x+\frac{nb}{c}\right)^2} \right\} \quad ; y \geq 0 \qquad \text{......(4.15)}$$

Now, to prove that $g_{(1:n)}(y)$ is a probability density function, we have to prove the following identity;

$$\int_0^\infty g_{(1:n)}(y)dx = 1$$

Taking Left Hand Side (LHS)

$$\int_0^\infty g_{(1:n)}(y)dx = \int_0^\infty \exp(-by)nb - \left\{ \frac{abn(n-1)}{c\left(y+\frac{na}{c}\right)} - \frac{an(n-1)}{c\left(y+\frac{na}{c}\right)^2} \right\} dx$$

$$= n\int_0^\infty b\exp(-by)dx - \int_0^\infty \left\{ \frac{abn(n-1)}{c\left(y+\frac{na}{c}\right)} - \frac{an(n-1)}{c\left(y+\frac{na}{c}\right)^2} \right\} \exp(-by)dy$$

Let $\left(y+\frac{na}{c}\right) = t$

then above expression becomes

$$= n - \frac{abn(n-1)}{c}\exp(nab|c)\left\{ \int_{na/c}^\infty \exp(-bt)t^{-1}dt - \frac{1}{b}\int_{na/c}^\infty \exp(-bt)t^{-2}dt \right\}$$

Using the relation

$$\int_x^\infty e^{-au}u^{-1}du = E_1(ax)$$

We get

$$\int_{x}^{\infty} g_{(1:n)}(y)dy = n-(n-1)=1$$

Now, the probability density function of the rth concomitant can be obtained by using the following formula as;

$$g_{(r:n)}(y) = \sum_{i=n=r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i} g_{(1:i)}(y)$$

where

$$g_{(1:i)}(y) = \exp(-by)\left\{ ib - \frac{abi(i-1)}{c\left(y+\frac{ia}{c}\right)} - \frac{ai(i-1)}{c\left(y+\frac{ia}{c}\right)^2} \right\} \qquad ......(4.16)$$

Thus,

$$g_{(r:n)}(y) = \sum_{i=n-r+1} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i} \exp(-by)\left\{ ib - \frac{abi(i-1)}{c\left(y+\frac{ia}{c}\right)} - \frac{ai(i-1)}{c\left(y+\frac{ia}{c}\right)^2} \right\}$$

Similarly we can obtain,

$$g_{(r:n)} = \sum_{i=n-r+1} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i} \exp(-by)\left\{ ib - \frac{abi(i-1)}{c\left(x+\frac{ia}{c}\right)} - \frac{ai(i-1)}{c\left(x+\frac{ia}{c}\right)^2} \right\}$$

4.6 Moments of Concomitants

Thus k^{th} moment about origin of the first concomitant i.e. of $Y_{[1:n]}$ is given by,

$$\mu^k_{y(1:n)} = \int_0^\infty y^k g_{(1:n)}(y)dy = \int_0^\infty y^k \exp(-by)\left\{ nb - \frac{abn(n-1}{c\left(y+\frac{na}{c}\right)} - \frac{an(n-1)}{c\left(y+\frac{na}{c}\right)^2} \right\}dx$$

Now using the following relation,

$$\int_0^\infty \exp(-pt)t^{\nu}dt = p^{-(\nu+1)}\Gamma(\nu+1) \qquad \mathrm{Re}(\nu) \geq -1, \quad \mathrm{Re}(p) \geq 0$$

we get

$$= n\frac{\Gamma(k+1)}{b^k} - \frac{abn(n-1)}{c}\left\{ \int_0^\infty \exp(-by)y^k\left(y+\frac{na}{c}\right)^{-1}dy - \frac{1}{b}\int_0^\infty \exp(-by)y^k\left(y+\frac{na}{c}\right)^{-2}dy \right\}$$

Integrating by parts and using the following relation,

$$\int_0^\infty \exp(-pt)t^{\nu}(t+\alpha)^{-1}dt = \Gamma(\nu+1)\alpha^{\nu}\exp(\alpha p)\Gamma(-\nu,\alpha p)$$

$$\{|\arg\alpha| < \pi, \qquad \mathrm{Re}(\nu) \geq 1, \qquad \mathrm{Re}(p) \geq 0\}$$

Here $d = \frac{na}{c}, p = b, t = y, \nu = k$

we get

$$\mu^k_{y(1:n)} = n\frac{\Gamma(k+1)}{b^k} - \left(\frac{na}{c}\right)^k (n-1)\Gamma(k+1)\exp\left(\frac{nab}{c}\right)\left(\Gamma(1-k,\frac{nab}{c}\right) \quad ...(4.19)$$

Now, the k^{th} moment about origin of $Y_{[r:n]}$ will be,

$$\mu^k_{y(r:n)} = \sum_{i=n-r+1}^{n}(-1)^{i-n+r-1}\binom{i-1}{n-r}\binom{n}{i}\mu^k_{(1:i)}$$

Where

$$\mu^k_{y(1:i)} = i\frac{\Gamma)k+1)}{b^k} - \left(\frac{ia}{c}\right)^k (i-1)\Gamma(k+1)\exp\left(\frac{iab}{c}\right)\left(\Gamma(1-k,\frac{iab}{c}\right)$$

Hence,

$$\mu^k_{y(r:n)} = \sum_{i=n-r+1}^{n}(-1)^{i-n+r-1}\binom{i-1}{n-r}\binom{n}{i}\left\{i\frac{\Gamma(k+1)}{b^k} - \left(\frac{ia}{c}\right)^k (i-1)\Gamma(k+1)\exp\left(\frac{iab}{c}\right)\Gamma\left(1-k,\frac{iab}{c}\right)\right\}$$

......(4.20)

Similarly, the k^{th} moment about origin of $X_{(r:n)}$ can be obtained as

$$\mu^k_{x(r:n)} = \sum_{i=n-r+1}^{n}(-1)^{i-n+r-1}\binom{i-1}{n-r}\binom{n}{i}\left\{i\frac{\Gamma(k+1)}{b^k} - \left(\frac{ib}{c}\right)^k (i-1)\Gamma(k+1)\exp\left(\frac{iab}{c}\right)\left(\Gamma(1-k,\frac{iab}{c}\right)\right\}$$

......(4.21)

Now, in particular for k=1 the mean of $y_{(r:n)}$ will be;

$$E(Y_{(r:n)}) = \mu^{1}_{y(r:n)} = \sum_{i=n-r+1}^{n}(-1)^{i-n+r-1}\binom{i-1}{n-r}\binom{n}{i}\left\{\frac{i}{b} - \frac{ia}{c}(i-1)\exp\left(\frac{iab}{c}\right)\Gamma\left(0,\frac{iab}{c}\right)\right\}$$

......(4.22)

The variance of Y(r:n) can be obtained as

$$V(Y_{(r:n)} = \mu^2_{y(r:n)} - \{\mu^1_{y(r:n)}$$

Where

$$\mu^2_{y(m:n)} = \sum_{i=n-r+1}^{n}(-1)^{i-n-r+1}\binom{i-1}{n-r}\binom{n}{i}\left\{\frac{2}{b^2} - 2\left(\frac{ia}{c}\right)^2 (i-1)\exp\left(\frac{iab}{c}\right)\Gamma(-1,\frac{iab}{c}\right\}$$

......(4.23)

Similarly, the mean of X(r:n) will be;

$$E\left(X_{(r:n)}\right)=\mu^{1}_{x(r:n)}=\sum_{i=n-r+1}^{n}(-1)^{i-n+r-1}\binom{i-1}{n-r}\binom{n}{i}\left\{\frac{i}{a}\cdot\frac{ib}{c}(i-1)\exp\left(\frac{iab}{c}\right)\Gamma\left(0,\frac{iab}{c}\right)\right\}$$

......(4.24)

Teh variance of $X_{(r:n)}$ can be obtained as

$$V(X_{(r:n)}=\mu^{2}_{x(r:n)}-\left\{\mu_{x(r:n)}\right\}^{2}$$

Where

$$\mu^{2}_{x(r:n)}=\sum_{i=n-r+1}^{n}(-1)^{i-n-r+1}\binom{i-1}{n-r}\binom{n}{i}\left\{\frac{2}{a^{2}}-2\left(\frac{ib}{c}\right)^{2}(i-1)\exp\left(\frac{iab}{c}\right)\Gamma(-1,\frac{iab}{c}\right\}$$

......(4.25)

It may be noted that;

$$\sum_{r=1}^{n}\mu^{1}_{y(r:n)}=n\left(\frac{1}{b}\right)=nE(Y) \qquad ......(4.26)$$

and

$$\sum_{r=1}^{n}\mu^{2}_{y(r:n)}=n\left(\frac{2}{b^{2}}\right)=nE(Y^{2}) \qquad ......(4.27)$$

4.7 Joint Distribution of Two Concomitants $Y_{(r:n)}$ and $Y_{(s:n)}$

Now, for the Gumbel's bivariate exponential distribution with probability density function (4.2) the joint distribution of two concomitants of order statistics $Y_{(r:n)}$ and $Y_{(s:n)}$ can be obtained by using the following relation;

$$f_{y(r:n),y(s:n)}(y_1 y_2) = g_{(r,s:n)}(y_1, y_2)$$

$$g_{(r,s:n)}(y_1 y_2) = \int_0^{\infty}\int_0^{x_2} f(y_1|x_1) f(y_2|x_2) f_{r,s:n}(x_1, x_2) dx_1 dx_2$$

Where $f_{r,s:n}(x_1 x_2)$ is the joint probability density function of r^{th} and s^{th} order statistics of X.

$$= \int_0^{\infty}\int_0^{x_2} [(b+cx_1)(a+cy_1)-c]\exp[-y_1(b+cx_1)][(b+cx_2)(a+cy_2)-c]\exp[-y_2(b+cx_2)]$$

$$C_{r,s:n} a^2 \sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{(i+j)}\binom{r-1}{i}\binom{s-r-1}{j}\exp[-(ia+sa-ra-ja)x_1]\exp[-(na-sa+ja+a)x_2]dx_1 dx_2$$

$$\left\{\int_0^{\infty}[(b+cx_2)(a+cy_2)-c]\exp[-(cy_2+na-sa+ja+a)x_2)\{I(x_2, y_1\}dx_2\right\}$$

......(4.28)

Where,

$$I(x_2 y_1) = \int_0^{x_2}[(b+cx_1)(a+cy_1)-c]\exp[-cy_1+ia+sa-ra-ra-ja)x_1]dx_1$$

Integrating by parts and simplifying the above expression, we get;

$$I(x_2, y_1) = \frac{c(a+cy_1)}{k(y_1)}\left\{-x_2\exp[-k(y_1)x_2] - \frac{1}{k(y_1)}\exp[-k(y_1)x_2 + \frac{1}{k(y_1)}\right\}$$
$$-\left\{[b(a+cy_1)-c]\frac{1}{k(y_1)}\{\exp[-k(y_1)x_2]-1\right\}$$

with $k(y_1) = cy_1 + ia + sa–ra–ja$

Putting the vlaue of $I(x_2, y_1)$ in equation (4.28), we get,

$$= C_{r,s:n}\exp[-b(y_1+y_2)]\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)(i+j)\binom{r-1}{i}\binom{s-r-1}{j}$$

$$\left\{\begin{array}{c}[(b+cx_2)(a+cy_2)]\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)(i+j)\binom{r-1}{i}\binom{s-r-1}{j}\\ \left\{\frac{c(a+cy_1)}{k(y_1)}\left\{-x_2\exp[-k(y_1)x_2]-\frac{1}{k(y_1)}\exp[-k(y_1)x_2]\frac{1}{k(y_1)}\right\}\right. \\ \left.\left\{[b(a+cy_1)-c]\frac{1}{k(y_1)}\{\exp[-k(y_1)x_2]-1\}\right\}\right\}\end{array}\right\}$$

Hence we get

$$g_{(r,s:n)}(y_1y_2)=C_{r,s:n}\exp(-b(y_1y_2))\sum_{i=0}^{r-1}\sum_{j=0}^{s-r-1}(-1)^{i+j}\binom{r-1}{j}\binom{s-r-1}{j}$$

$$\frac{(a+cy_1)(a+cy_2)}{k(y_1)}\left\{c^2\left\{\frac{1}{k(y_1)k^2(y_2)}-\frac{2}{k(y_1)k(y_2)^3}-\frac{1}{k(y_1)(k_1(y_1)+k(y_2)}\right\}\right\}$$

$$+b^2c^2\frac{k(y_1)}{k(y_2)(k(y_1)+k(y_2)}$$

$$+\frac{c^2(a+(y_1)}{k(y_1)}\left(\frac{1}{(k(y_1)+k(y_2))^2}+\frac{1}{k(y_1)(k(y_1)+k(y_2)}-\frac{1}{k(y_1)k(y_2)}\right)$$

$$-\frac{bc((a+cy_1)}{k(y_1)}.\frac{k(y_1)}{k(y_2)(k(y_1)+k(y_2)}+\frac{c^2(a+(y_2)}{k(y_1)}-\left(\frac{1}{(k(y_1)+k(y_2)^2}-\frac{1}{k^2(y_2)}\right)$$

$$-\frac{bc(a+(y_2)}{k(y_1)}.\frac{k(y_1)}{k(y_2)(k(y_1)+k(y_2)}+\frac{c^2}{k(y_2)(k(y_1)+k(y_2))}$$

with $\quad k(y_1) = cy_1 + ia + sa - ra - ja$ and

$$k(y_2) = cy_2 + na - sa + ja + a$$

CHAPTER - 5

BAYESIAN ESTIMATION OF SYSTEM RELIABILITY

5.0 INTRODUCTION

In the context of modern technology and its future developments system reliability assessment is of paramount importance. The objectives of a system engineer include the study of various reliability characteristics of the system for improving upon its performance. These characteristics can be better analyses if apart from the observed life data the environmental experience of the variations in the reliability parameters of the system is also taken into consideration. Bayesian analysis combines the prior belief about the life time parameters with experimental data. The studies by Balagurusamy (1984), Sinha (1986), Mann et al (1974) & Kapur & Lambason (1977) used classical and Bayesian inference technique in case of some important life time. In many practical situations the operational data on the complete system are either limited or non-existent. In such cases, one makes use of the life time data recorded on its components. The studies by Kaplan et al (1989). A apostolicism

(1990) Martz and Waller (1982) reviewed the relevant literature on probability safely assessment (PSA) of a system and laid out a language conceptual frame work and methodology for dealing with such situations. Kaplan et al (1989) analysed various probability curves useful for expressing the degree of confidence about the complete system reliability and the way in which Bayes theorem updates prior probability curves. Highlighting the importance of posterior estimates for reliability characteristics of various system networks. Sharma and Bhutani (1992a) presented Bayesian reliability analysis using the truncated failure information.

For improving the performance of a system maintenance is used by reliability practitioners maintainability of a maintained system is as important as reliability itself for an engineer. The concept of availability combines both reliability and main ability. Inferences on system availability have been studied in detail in the literature both from classical and Bayes view point. Most of these studies used simple probabilisitcs reasoning for analysing availability of various systems. Further, a system may vary from simple to a vary complex one. One approach for analysing such a system is to decompose it into subsystems of convenient sizes, each representing a specific function, reliability and availability of susbsystems

are then estimated and combined to determine the reliability of the entire system using certain probability laws. In the light of above discussion the present Chapter develop Bayes estimators of the system reliability and available in case of some important and basic system configuration.

5.1. STATISTICAL ASSUMPTIONS

For performing the statistical analysis on the system reliability characteristics

(1) It is assumed that the failure time distribution for each component is negative exponential with p.d.f.

$$f(x;\theta) = \theta e^{-\theta x}; x, \theta > 0 \qquad \text{.......(5.1)}$$

Therefore θ and $\frac{1}{\theta}$ are the failure rate and mean time to system failure (MTSF) respectively.

(2) It is assumed that the repair time distribution for each component is also negative exponential with p.d.f.

$$f(y;\theta') = \theta' e^{-\theta' y}; y, \theta > 0 \qquad \text{.......(5.2)}$$

Here θ' and $\frac{1}{\theta'}$ stand for repair rate and mean time to system repair (MTTR) respectively.

(3) The reliability function for the system or component say R(t) may be defined as

$$R(t) = P[x > t] = e^{-\theta t}; t \geq 0 \qquad ....(5.3)$$

Where t is called the mission time

(4) The system or component availability is the probability that the system is operative in the long run and may be expressed as

$$\text{Availability} = \frac{\text{MTSF}}{\text{MTSF} + \text{MTTR}}$$

$$= \frac{\frac{1}{\theta}}{\frac{1}{\theta} + \frac{1}{\theta'}} = \frac{\theta'}{\theta + \theta'} \qquad ....(5.4)$$

5.2. RELIABILITY AND AVAILABILITY FOR SOME BASIC SYSTEM CONFIGURATIONS

Various basic system configurations have been discussed earlier. It is notable that using basic probability laws one can express reliability of the whole system in terms of the rehabilitates of its components. Almost on the same lines, one can obtain expressions for system availability in terms of component availability. Here, it has been assumed that each system configuration consists of independent and identical

components. Denoting the reliability and availability functions for series-parallel, parallel-series and a k-out-of-m system by $R_{sp}(t), A_{sp}, R_{ps}(t), A_{ps}, R_{km}(t)$ and A_{km} respectively, some well known results are stated below.

5.2.1 SERIES-PARALLEL SYSTEM

This system consists of n-subsystems arranged in series and each subsystem has m-parallel components.

For this system

$$R_{sp}(t) = \left(1 - \{1 - R(t)\}^{m}\right)^{n} \qquad \text{.......(5.5)}$$

and

$$A_{sp} = \left(1 - \{1 - A\}^{m}\right)^{n} \qquad \text{.......(5.6)}$$

Considering $(1-z)^n = \sum_{\alpha=0}^{n} (-1)^{\alpha} z^{\alpha} \binom{n}{\alpha}$,

Where $\binom{n}{\alpha} = 0, \alpha > n$

the equations (5.5) and (5.6) may also be rewritten as

$$R_{sp}(t) = \sum_{\alpha=0}^{n} \sum_{\beta=0}^{m\alpha} (-1)^{\alpha+\beta} \binom{n}{\alpha} \cdot \binom{m\alpha}{\beta} [R(t)]^{\beta}, t > 0 \qquad \text{.......(5.7)}$$

and

$$A_{sp} = \sum_{\alpha=0}^{n} \sum_{\beta=0}^{m\alpha} (-1)^{\alpha+\beta} \binom{n}{\alpha} \binom{m\alpha}{\beta} A^{\beta} \qquad \text{.......(5.8)}$$

5.2.2 PARALLEL SERIES SYSTEM

In this configuration there are n-parallel subsystems each having m-component arranged in series. Here,

$$R_{ps}(t) = 1 - [1 - \{R(t)\}^m]^n \qquad \text{.......(5.9)}$$

or

$$R_{ps}(t) = \sum_{\alpha=1}^{n} (-1)^{\alpha-1} \binom{n}{\alpha} [R(t)]^{m\alpha}, t > 0 \qquad \text{......(5.10)}$$

and

$$A_{ps} = 1 - [1 - A^m]^n \qquad \text{......(5.11)}$$

or

$$A_{ps} = \sum_{\alpha=1}^{n} (-1)^{\alpha-1} \binom{n}{\alpha} . A^{m\alpha} \qquad \text{......(5.12)}$$

5.2.3 K-OUT-OF-M SYSTEM

With simple probabilistic reasoning, the reliability for this system is given by

$$R_{km}(t) = \sum_{u=k}^{m} \binom{m}{u} [R(t)]^u [1 - R(t)]^{m-u} \qquad \text{......(5.13)}$$

or

$$R_{km}(t) = \sum_{u=k}^{m} \sum_{v=0}^{m-u} (-1)^v \binom{m}{u} \binom{m-u}{v} [R(t)]^{u+v} \qquad \text{......(5.14)}$$

and

$$A_{km} = \sum_{u=k}^{m} \binom{m}{u} A^u (1 - A)^{m-u} \qquad \text{.....(5.15)}$$

5.3 DEVELOPMENT OF POSTERIOR DISTRIBUTIONS

Now, since, the failure time distribution for the components is negative exponential with failure rate μ therefore, fora total testing time T, the probability of observing r failures will be given by the poison distribution with a p.m.f.

$$P(r|\mu) = \frac{e^{-\mu T}(\mu T)^r}{r!}; r = 0,1,2..... \qquad ......(5.16)$$

Similarly, the number of component repair(s) performed in the me (0,1') will also follow the poison distribution with p.m.f.

$$P(s|\lambda) = \frac{e^{-\lambda T}(\lambda T)^2}{s!}; s = 0,1,2... \qquad ......(5.17)$$

Now, the posterior distribution of μ in respect of the prior in (5.5) given that r failures have been observed in the time interval (0, T) can be obtained as

$$\Pi(\mu|r) = \frac{P(r|\mu)g(\mu)}{\int_0^\infty P(r|\mu)g(\mu)d\mu}$$

$$= \frac{(T+1)^{r+a} e^{-(T+1)\mu} \mu^{r+a-1}}{\Gamma(r+a)}; \mu, (T+1), (r+a) > 0 \qquad ......(5.18)$$

Which is Gamma density with parameters (T+1) and (r+a).

Similarly, the posterior distribution of λ in respect of the prior (5.6) can be put as

$$\Pi(\lambda|s) = \frac{P(s|\lambda)g(\lambda)}{\int_0^\infty P(s|\lambda)g(\lambda)d\lambda}$$

$$= \frac{(T+1)^{s+b} e^{-(T+1)\lambda} \lambda^{s+b-1}}{\Gamma(s+b)}; \lambda, (T+1), (s+b) > 0 \quad \text{......(5.19)}$$

The distribution in (5.19) is also a gamma density with parameters $(T+1)$ and $(s+b)$.

Now, on using the respective posterior distributions of μ and λ in (5.18) and (5.19) and using the transformation $A = \frac{\lambda}{\lambda+\mu}$ in (5.8) the posterior distribution of A given r and s comes out to be a beta distribution with p.d.f.

$$f(A|r,s) = \frac{1}{B(s+b, r+a)} A^{s+b-1}(1-A)^{r+a-1}; 0 < a < 1, (s+b), (r+a) > 0 \quad \text{......(5.20)}$$

5.4. BAYESIAN ANALYSIS

5.4.1. BAYES ESTIMATES OF RELIABILITY FUNCTIONS FOR SERIES-PARALLEL, PARALLEL-SERIES AND K-OUT-OF-M SYSTEMS

For series-parallel, parallel-series using (5.5), (5.7) and (5.19), the Bayes point estimate of the reliability function for a series-parallel system can be defined as:

$$R^*_{sp}(t) = E\left[R_{sp}(t)|r\right]$$

$$= R_{sp}(t)\pi(t)\pi(\mu|r)d\mu$$

$$= \sum_{\alpha=0}^{m}\sum_{\beta=0}^{m\alpha}(-1)^{\alpha+\beta}\binom{n}{\alpha}\binom{m\alpha}{\beta}\frac{(T+1)^{r+a}}{\Gamma(r+a)}\int_0^\infty \mu^{r+a-1}e^{-(T+t\beta+1)}d\mu$$

$$= \sum_{\alpha=0}^{m}\sum_{\beta=0}^{m\alpha}(-1)^{\alpha+\beta}\binom{n}{\alpha}\binom{m\alpha}{\beta}\frac{1}{\left(1+\frac{T\beta}{T+1}\right)^{r+a}} \qquad \text{......(5.21)}$$

and on using (5.5), (15.10) and (5.19), the Bayes point estimate of the reliability function fora parallel-series system becomes:

$$R^*_{ps}(t) = E\left[R_{ps}(t)|r\right]$$

$$= \sum_{\alpha=0}^{m}(-1)^{\alpha+1}\binom{n}{\alpha}\frac{(T+1)^{r+a}}{\Gamma(r+a)}\int_0^\infty \mu^{r+a-1}e^{-(T+m\alpha t+1)}\mu_{d\mu}$$

$$= \sum_{\alpha=1}^{n}(-1)^{\alpha+1}\binom{n}{\alpha}\left(\frac{1}{1+\frac{m\alpha t}{T+1}}\right)^{r+a} \qquad \text{......(5.22)}$$

Similarly, on using (5.5), (5.14) and (5.19), the Bayes point estimate of the reliability function for a k-out-of-m system can be put as:

$$R^*_{km}(t)l = E\left[R_{km}(t)|r\right]$$

$$= \sum_{u=k}^{m}\sum_{v=0}^{m-u}(-1)^{v}\binom{m}{u}\binom{m-u}{v}\frac{(T+1)^{r+a}}{\Gamma(r+a)}\int_0^\infty \mu^{r+a-1}e^{-(T+(u+v)+1}d\mu$$

$$= \sum_{u=r}^{m}\sum_{v=0}^{m-u}(-1)^{v}\binom{m}{u}\binom{m-u}{v}\frac{1}{\left[1+\frac{(u+v)t}{T+1}\right]^{r+a}} \qquad \text{......(5.23)}$$

5.4.2. BAYES ESTIMATES OF AVAILABILITY OF A SERIES-PARALLEL, PARALLEL-SERIES AND K-OUT-OF-M SYSTEMS

Using (5.6), (5.8) and (5.20), the Bayes point estimate of availability of a series-parallel system can be obtained as:

$$A_{sp}^{0} = E\left[A_{sp} \mid r,s\right]$$

$$= \int_0^1 A_{sp} f(a \mid r,s) dA$$

$$= \sum_{\alpha=0}^{n} \sum_{\beta=0}^{m\alpha} (-1)^{\alpha+\beta} \binom{n}{\alpha} \binom{m\alpha}{\beta} \frac{1}{B(s+b,r+a)} \int_0^1 A^{s+b+\beta-1} (1-A)^{r+a-1} dA$$

$$= \sum_{\alpha=0}^{n} \sum_{\beta=0}^{m\alpha} (-1)^{\alpha+\beta} \binom{n}{\alpha} \binom{m\alpha}{\beta} \frac{B(s+b+\beta,r+a)}{B(s+b,r+a)} \qquad \text{......(5.24)}$$

Similarly, on using (5.6), (5.12), (5.20), the Bayes estimate of availability of a parallel-series system can be found as:

$$A_{ps}^{0} = E\left[A_{ps} \mid r,s\right]$$

$$= \sum_{\alpha=1}^{n} (-1)^{\alpha-1} \binom{n}{\alpha} \frac{1}{B(s+b,r+a)} \int_0^1 A^{s+b+m\alpha-1} (1-A)^{r+a-1} dA\}$$

$$= \sum_{\alpha=1}^{n} (-1)^{\alpha-1} \binom{n}{\alpha} \frac{B(s+b+m\alpha,r+a)}{B(s+b,r+a)} \qquad \text{......(5.25)}$$

Also, on using (15.6), (5.15) and (5.20), the Bayes point estimate of availability of a k-out-of-m system becomes

$$A_{km}^{0} = E\left[A_{km} \mid r,s\right]$$

$$= \sum_{u=k}^{m} \binom{m}{u} \frac{1}{B(s+b,r+a)} \int_0^1 A^{s+b+u-1} (1-A)^{r+a+m-u-1} dA$$

$$= \sum_{u=k}^{m} \binom{m}{u} \frac{B(s+b+u,r+a+m-u)}{B(s+b,r+a)} \qquad \text{......(5.26)}$$

5.4.3 An Example

For the purpose of studying the behaviour of posterior or Bayes estimates of reliability and availability functions of series-parallel, parallel-series and a k-out-of-m systems, we consider an example with m=n=5, k=3 and mission time t=30 units. Further also let r=3 and s=2 be the recorded failure and repair information in the time interval (0,100). Using (5.21), (5.22), (5.23), 5.24), (5.25), (5.25) and (5.26), the Bayes estimates of reliability and availability for varying mean failure rate 'a' and fixed mean repair rate 'b' have been tabulated in table-I and II.

Table - I

Bayes point estimate of reliability function for m=n=5, r=k=3 and varying mean failure rate 'a' and fixed T = 100 units.

S.No.	A	$R^*sp(t)$	$R^*ps(t)$	$R^*km(t)$
1.	1	0.493	0.100	0.285
2.	2	0.341	0.044	0.178
3.	3	0.221	0.019	0.107
4.	4	0.136	0.007	0.063
5.	5	0.080	0.003	0.036

Table - II

Bayes point estimate of availability for m=n=5, r=k=3, s=2

and varying mean failure rate 'a' and fixed repair rate 'b'

S.No.	a	b	A^*_{sp}	A^*_{ps}	A^*_{km}
1.	1	1	0.955	0.1654	0.392
2.	2	1	0.910	0.105	0.311
3.	3	1	0.888	0.069	0.249
4.	4	1	0.819	0.046	0.203

5.4.4 CONCLUDING REMARKS

In many practical situations, it is important to estimate the reliability and availability of the static system models when some operational experiences with the complete system are available as experimental data. In the present chapter, time truncated failure and repair information have been used to update the analysis regarding the reliability and availability of such system.

Table-I reveal that the Bayes estimates of $R_{sp}(t)$, $R_{ps}(t)$ and $R_{km}(t)$ tend to be lower as the mean failure rate 'a' increases. Also for fixed mean failure rate 'b' the Table II shows that the Bayes estimates of A_{sp} and A_{km} tend to be lower as mean failure rate 'a' increases, rate 'a' increases.

5.5. OPTIMIZATION OF SYSTEM AVAILABILITY IN THE BAYESIAN FRAMEWORK

5.5.1. Introduction

Optimization of reliability characteristics subject to constraints on the number of components in static system models has been the main concern of reliability engineers. Reliability and availability are two important characteristics of a system or device. For a system, availability is defined as the ratio of mean time between failure (MTBF) to the mean time between failure (MTBF) plus mean time to repair (MTTR). Therefore, both the reliability and maintainability aspects are considered for defining availability. Most of the reliability characteristics of various lifetime distribution have been analysed in respect of their~ statistical properties. However, inferences on the system availability are not many in literature. Gray and Lewis (1967), Masters and Lewis [1987], and Masters, Lewis and Kolarik (1992) have touched this aspect to some extent. They developed confidence interval for availability after separately establishing confidence intervals for MTBF and MTTR, Further, with the advancement in technology in present time, it is reasonable to assume random variations in the parameters of failure time and repair time distribution due to environmental conditions. Thus, the Bayesina analysis

of the system availability become important which combines past experience with experimental data. In view of the above, the present section deals with the development of Bayesian tolerance limits for the availability of some basic system models, Subject to cosy constraints, these limits may be analysed to obtain tile optimum number of components or subsystems in a system models with a preassigned confidence coefficient. Time truncated sample information have been used in analysis.

5.5.2. Bayesian tolerance limits for system availability

The Bayesian tolerance limits for system availability (A_s) a~needed to attain at least p availability, with at least confidence coefficient γ, may be expressed as :

$$P[A_s \geq p] \geq \gamma; 0 < p, \gamma < 1 \qquad \text{......(5.27)}$$

Now, using (5.27) and the results in sections 5.2, the Bayesian admissible standards expressed in terms of p and y, these limits may be used to find the optimum number of components or subsystems in different system models.

5.5.3. (I) Series-parallel system

For a series-parallel system, at least (100)% Bayesian tolerance~limits for A_5 in (5.6) can be obtained from (5.27), i.e. using: (5.6) in (5.27), one gets,

$$P[\{1-(1-A)^m\}^n \geq p] \geq \gamma$$

Or $P[A \geq \{1-(1-p^{1/n})^{1/m}\}] \geq \gamma$

Or $P[A \leq \{1-(1-p^{1/n})^{1/m}\}] \geq 1-\gamma$

Now, on using the posterior distribution of A in (5.20) we have

$$\frac{1}{B(s+b,r+a)} \int_0^{1-(1-p^{1/n})^{1/m}} A^{s+b-1}(1-A)^{r+a-1} dA \leq 1-\gamma$$

$$\text{Or } I_{z_1}(s+b,r+a) \leq 1-\gamma \qquad \text{......(5.28)}$$

Where $z_1 = 1-(1-p^{1/n})^{1/m}$ and $I_{z_1}(s+b,r+a)$ is the well known incomplete beta ratio.

For attaining at least p availability with admissible γ, the inequality (5.28) can be used to optimize m(n) with known n(m).

5.5.4. (II) Parallel-series system

Similarly on using (5.11) in (5.27), one gets,

$$P\left[\{1-(1-A^n)^m\} \geq p\right] \geq \gamma$$

Or $P\left[A \geq \{1-(1-p)^{1/m}\}^{1/n}\right] \geq \gamma$

Or $P\left[A \le \{1-(1-p)^{1/m}\}^{1/n}\right] \le 1-\gamma$

Or $\dfrac{1}{B(s+b,r+a)} \displaystyle\int_0^{[1-(1-p)^{1/m}]^{1/n}} A^{s+b-1}(1-A)^{r+a-1}dA \le 1-\gamma$

Or $I_{Z_2}(s+b,r+a) \le 1-\gamma$ ......(5.29)

Inequality (5.29) can also be analysed as in the series-parallel case.

5.5.5. (III) K-out-of-m System

On substituting in, that limits for this system becomes some arbitrarily chosen values of p and g have been summarized in table III and IV. These values can be analysed to optimize the values of m and n for pre-assigned values of p and y in a series, parallel, series-parallel and parallel-series system models. Further, it is to be noted here that the results for the different system models in talbe III and IV show that an additional component in series adversely affects the values of p and However, these values attain a reasonable standard when an additional component is placed in a parallel system models. Similarly, the optimization of availability in the case of a k-out-of-m system can be achieved by analysing the relevant inequalities.

Table III

Bayesian tolerance limits for availability of a series and parallel system

Series System			Series System		
p	γ	n	p	γ	N
0.18	0.34	2	0.60	0.60	3
0.45	0.34	1	0.80	0.60	4
0.10	0.40	2	0.90	0.60	6
0.10	0.20	3	0.70	0.50	3
0.20	0.10	3	0.70	0.85	5
0.20	0.20	2	0.70	0.90	6

Table IV

Bayesian tolerance limits for availability of a series - parallel and

Parallel-series system

Series-Parallel System				Parallel-Series System			
p	γ	m	n	p	γ	m	n
0.40	0.60	3	2	0.50	0.50	5	2
0.30	0.60	3	3	0.70	0.50	8	2
0.50	0.40	4	5	0.60	0.54	7	2
0.50	0.50	4	4	0.60	0.60	8	2
0.54	0.70	4	2	0.40	0.27	10	4
0.35	0.70	4	3	0.40	0.40	16	4

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